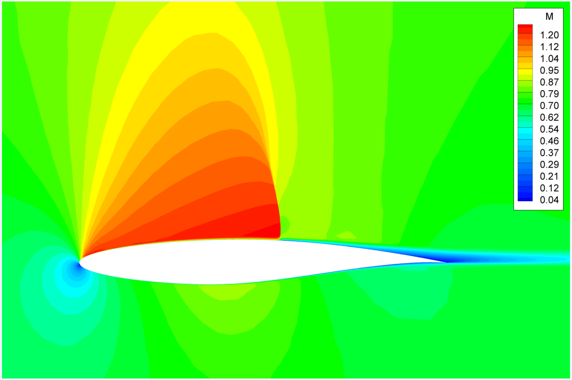
Aerodynamics I

Compressible flow past an airfoil



transonic flow past the RAE-2822 airfoil ($M = 0.73$, $Re = 6.5 \times 10^6$, $\alpha = 3.19^\circ$)

Potential equation in compressible flows

Full potential theory

Let us introduce a velocity potential (no vorticity is present $\nabla \times \mathbf{v} = 0$):

$$\mathbf{v} = \nabla \Phi \quad \xrightarrow{\text{2D}} \quad u = \frac{\partial \Phi}{\partial x} \qquad v = \frac{\partial \Phi}{\partial y} \tag{1.1}$$

Continuity equation can be transformed:

$$\nabla \cdot (\rho\, \mathbf{v}) = \rho \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \rho = 0 \quad \rightarrow \quad \rho \nabla^2 \Phi + \nabla \Phi \cdot \nabla \rho = 0 \tag{1.2}$$

From momentum equation:

$$-\frac{1}{\rho} \nabla p = \mathbf{v} \cdot \nabla \mathbf{v} = \nabla \left(\frac{V^2}{2} \right) + (\nabla \times \mathbf{v}) \times \mathbf{v} \tag{1.3}$$

Since vorticity $\nabla \times \mathbf{v}$ is zero :

$$(\nabla \times \mathbf{v}) \times \mathbf{v} = 0 \quad \rightarrow \quad dp = -\frac{\rho}{2}\, d\left(V^2\right) = -\frac{\rho}{2}\, d\left(\nabla \Phi \cdot \nabla \Phi\right) \tag{1.4}$$

Using equation for speed of sound (izentropic relation):

$$dp = c^2\, d\rho \quad \rightarrow \quad d\rho = -\frac{\rho}{2\, c^2}\, d\left(\nabla \Phi \cdot \nabla \Phi\right) \tag{1.5}$$

Full potential theory

Gradient of density can be obtained fro equation:

$$\nabla\rho = -\frac{\rho}{2\,c^2}\,\nabla\left(\nabla\Phi\cdot\nabla\Phi\right) = -\frac{\rho}{c^2}\,\nabla\Phi\cdot\nabla\nabla\Phi \qquad (1.6)$$

for 2D:

$$\frac{\partial\rho}{\partial x} = -\frac{\rho}{c^2}\left[\frac{\partial\Phi}{\partial x}\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial\Phi}{\partial y}\frac{\partial^2\Phi}{\partial x\partial y}\right]$$

$$\frac{\partial\rho}{\partial y} = -\frac{\rho}{c^2}\left[\frac{\partial\Phi}{\partial x}\frac{\partial^2\Phi}{\partial x\partial y} + \frac{\partial\Phi}{\partial y}\frac{\partial^2\Phi}{\partial y^2}\right] \qquad (1.7)$$

$$(1.8)$$

after substituting into (1.2):

$$\rho\,\nabla^2\Phi - \frac{\rho}{c^2}\,\nabla\Phi\cdot\left(\nabla\Phi\cdot\nabla\nabla\Phi\right) = 0 \qquad (1.9)$$

for 2D:

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} - \frac{1}{c^2}\left[\left(\frac{\partial\Phi}{\partial x}\right)^2\frac{\partial^2\Phi}{\partial x^2} + \left(\frac{\partial\Phi}{\partial y}\right)^2\frac{\partial^2\Phi}{\partial y^2} + 2\,\frac{\partial\Phi}{\partial x}\frac{\partial\Phi}{\partial y}\frac{\partial^2\Phi}{\partial x\partial y}\right] = 0$$

$$(1.10)$$

Full potential theory

After transformation equation (1.10) takes following form:

$$\left[1-\frac{1}{c^2}\left(\frac{\partial\Phi}{\partial x}\right)^2\right]\frac{\partial^2\Phi}{\partial x^2}+\left[1-\frac{1}{c^2}\left(\frac{\partial\Phi}{\partial y}\right)^2\right]\frac{\partial^2\Phi}{\partial y^2}=\frac{2}{c^2}\frac{\partial\Phi}{\partial x}\frac{\partial\Phi}{\partial y}\frac{\partial^2\Phi}{\partial x\partial y} \tag{1.11}$$

Using energy integral the relation for speed of sound can be obtained:

$$c^2=c_0^2-\frac{k-1}{2}\left[\left(\frac{\partial\Phi}{\partial x}\right)^2+\left(\frac{\partial\Phi}{\partial y}\right)^2\right] \tag{1.12}$$

Equations (1.11) and (1.12) are equation of full potential theory. They allow to simulate compressible flows with weak (izentropic relations are used!) shock waves.

Linearized equations of small disturbances potential

Equations (1.11) and (1.12) can be further simplified by introducing potential of small disturbances and using linearization.

$$\mathbf{v} = \mathbf{v}_\infty + \widetilde{\mathbf{v}} \tag{1.13}$$

where:

$$\mathbf{v} = [\,u, v\,]^\top, \quad \mathbf{v}_\infty = [\,V_\infty, 0\,]^\top, \quad \widetilde{\mathbf{v}} = [\,\widetilde{u}, \widetilde{v}\,]^\top$$

Let us define potential of disturbances ϕ :

$$\widetilde{\mathbf{v}} = \nabla\phi \quad \rightarrow \quad \Phi = V_\infty\, x + \phi \tag{1.14}$$

After substituting into equation (1.11):

$$\left[c^2 - \left(V_\infty + \frac{\partial\phi}{\partial x}\right)^2\right]\frac{\partial^2\phi}{\partial x^2} + \left[c^2 - \left(\frac{\partial\phi}{\partial y}\right)^2\right]\frac{\partial^2\phi}{\partial y^2} = 2\left(V_\infty + \frac{\partial\phi}{\partial x}\right)\frac{\partial\phi}{\partial y}\,\frac{\partial^2\phi}{\partial x\partial y} \tag{1.15}$$

Linearized equations of small disturbances potential

In order to simplify the (1.15) it is convenient to use form based on velocity not the potential:

$$\left[c^2 - \left(V_\infty + \tilde{u}\right)^2\right]\frac{\partial \tilde{u}}{\partial x} + \left[c^2 - \tilde{v}^2\right]\frac{\partial \tilde{v}}{\partial y} = 2\left(V_\infty + \tilde{u}\right)\tilde{v}\,\frac{\partial \tilde{u}}{\partial y} \tag{1.16}$$

Equation (1.12) takes form:

$$c^2 = c_\infty^2 - \frac{k-1}{2}\left(2V_\infty \tilde{u} + \tilde{u}^2 + \tilde{v}^2\right) \tag{1.17}$$

After substitution (1.17) into (1.16) and transformation:

$$\begin{aligned} \left(1 - M_\infty^2\right)\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} = {} & M_\infty^2\left[(k+1)\frac{\tilde{u}}{V_\infty} + \frac{k+1}{2}\frac{\tilde{u}^2}{V_\infty^2} + \frac{k-1}{2}\frac{\tilde{v}^2}{V_\infty^2}\right]\frac{\partial \tilde{u}}{\partial x} \\ & + M_\infty^2\left[(k+1)\frac{\tilde{v}}{V_\infty} + \frac{k+1}{2}\frac{\tilde{v}^2}{V_\infty^2} + \frac{k-1}{2}\frac{\tilde{u}^2}{V_\infty^2}\right]\frac{\partial \tilde{v}}{\partial y} \\ & + M_\infty^2\left[\frac{\tilde{v}}{V_\infty}\left(1 + \frac{\tilde{u}}{V_\infty}\right)\left(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial x}\right)\right] \end{aligned} \tag{1.18}$$

Linearized equations of small disturbances potential

Assuming small disturbances (valid for thin airfoils at small angles of attack):

$$\frac{\tilde{u}}{V_\infty} \ll 1 \qquad \frac{\tilde{v}}{V_\infty} \ll 1 \tag{1.19}$$

Equation (1.18) can be transformed to a form where only linear terms are present (it is assumed that it is valid for $M_\infty < 0.8$ and $1.2 < M_\infty < 5$):

$$\left(1 - M_\infty^2\right) \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} = 0 \tag{1.20}$$

After substitution of the potential function:

$$\left(1 - M_\infty^2\right) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{1.21}$$

This equation is elliptic for $M_\infty < 1$ and hyperbolic for $M_\infty > 1$.

Pressure coefficient for linearized equations

Pressure coefficient is defined by:

$$C_p \equiv \frac{p - p_\infty}{q_\infty} \quad \text{gdzie:} \quad q_\infty = \frac{\rho_\infty V_\infty^2}{2} \tag{1.22}$$

Equation (1.22) can be transformed using:

$$q_\infty = \frac{\rho_\infty V_\infty^2}{2} = \frac{k\, p_\infty}{2} \left(\frac{\rho_\infty}{k\, p_\infty} \right) V_\infty^2 = \frac{k}{2}\, p_\infty\, M_\infty^2 \tag{1.23}$$

The formula for pressure coefficients takes then form:

$$C_p = \frac{2}{k\, M_\infty^2} \left(\frac{p}{p_\infty} - 1 \right) \tag{1.24}$$

For simplifications we will use izentropic relation::

$$\frac{p}{p_\infty} = \left(\frac{T}{T_\infty} \right)^{\frac{k}{k-1}} \tag{1.25}$$

Pressure coefficient for linearized equations

From eneregy integral:

$$\frac{V^2}{2} + c_p T = \frac{V_\infty^2}{2} + c_p T_\infty \quad \rightarrow \quad T - T_\infty = \frac{k-1}{2\,k\,R}\left(V_\infty^2 - V^2\right) \tag{1.26}$$

Using definition of the disturbance velocity (1.13) we can obtain:

$$\frac{T}{T_\infty} = 1 - \frac{k-1}{2}\,M_\infty^2\left(\frac{2\tilde{u}}{V_\infty} + \frac{\tilde{u}^2 + \tilde{v}^2}{V_\infty^2}\right) \tag{1.27}$$

After substituting:

$$\frac{p}{p_\infty} = \left[1 - \frac{k-1}{2}\,M_\infty^2\left(\frac{2\tilde{u}}{V_\infty} + \frac{\tilde{u}^2 + \tilde{v}^2}{V_\infty^2}\right)\right]^{\frac{k}{k-1}} \tag{1.28}$$

It can be simplified by expanding using Taylor series and discarding higher order terms:

$$\frac{p}{p_\infty} = (1-r)^{\frac{k}{k-1}} = 1 - \frac{k}{k-1} r + \ldots \tag{1.29}$$

$$\frac{p}{p_\infty} \approx 1 - \frac{k}{2}\,M_\infty^2\left(\frac{2\tilde{u}}{V_\infty} + \frac{\tilde{u}^2 + \tilde{v}^2}{V_\infty^2}\right) \tag{1.30}$$

Pressure coefficient for linearized equations

Substituting (1.30) inot (1.24):

$$C_p = -\frac{2\tilde{u}}{V_\infty} - \frac{\tilde{u}^2 + \tilde{v}^2}{V_\infty^2} \tag{1.31}$$

For small disturbances (1.19):

$$\frac{\tilde{u}^2 + \tilde{v}^2}{V_\infty^2} \ll \frac{2\tilde{u}}{V_\infty} \tag{1.32}$$

Finally, we can obtain the simplified relation for pressure coefficient:

$$C_p = -\frac{2\tilde{u}}{V_\infty} \tag{1.33}$$

The relations are valid for both, subsonic and supersonic flows.

Prandtl–Glauert rule

Let us consider subsonic flow ($M_\infty < 1$) which is described by linearized equations of small disturbances potential. We can defined the coefficient:

$$\beta \equiv \sqrt{1 - M_\infty^2} \tag{1.34}$$

equation (1.21) can be then written in form:

$$\beta^2 \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{1.35}$$

This equation can be transformed to Laplace equation by transformation applied to the reference coordinates, e.g.:

$$\xi = x \qquad \eta = \beta y \quad \rightarrow \quad \frac{\partial}{\partial x^2} = \frac{\partial}{\partial \xi^2} \qquad \frac{\partial}{\partial y^2} = \beta^2 \frac{\partial}{\partial \eta^2} \tag{1.36}$$

After substituting into (1.35):

$$\beta^2 \frac{\partial^2 \varphi}{\partial \xi^2} + \beta^2 \frac{\partial^2 \varphi}{\partial \eta^2} = 0 \quad \rightarrow \quad \frac{\partial^2 \varphi}{\partial \xi^2} + \frac{\partial^2 \varphi}{\partial \eta^2} = 0 \tag{1.37}$$

Prandtl–Glauert rule

What is a difference between ϕ and φ ?

Let us assume that the boundary of the airfoil on the plane $x - y$ is defined by function $y = f(x)$. Boundary condition for the airfoil (normal velocity is equal zero) then can be defined:

$$\frac{d}{dx} f(x) = \frac{\tilde{v}}{V_\infty + \tilde{u}} \approx \frac{\tilde{v}}{V_\infty} \quad \rightarrow \quad \tilde{v} = \frac{\partial \phi}{\partial y} = V_\infty \frac{d}{dx} f(x) \tag{1.38}$$

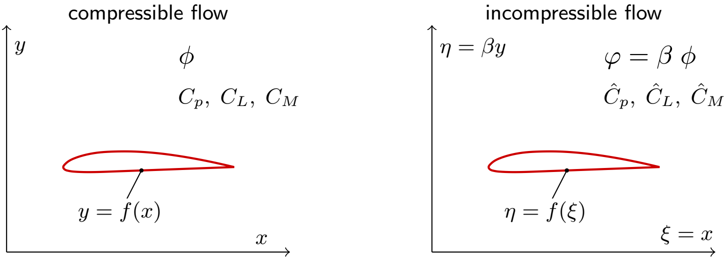
In similar way for $\xi - \eta$:

$$\frac{d}{d\xi} \hat{f}(\xi) = \frac{\hat{v}}{V_\infty + \hat{u}} \approx \frac{\hat{v}}{V_\infty} \quad \rightarrow \quad \hat{v} = \frac{\partial \varphi}{\partial \eta} = V_\infty \frac{d}{d\xi} \hat{f}(\xi) \tag{1.39}$$

Assuming that airfoils in $x - y$ and $\xi - \eta$ are similar then $\hat{f} = f$.

$$\frac{\partial \varphi}{\partial \eta} = \frac{1}{\beta} \frac{\partial \varphi}{\partial y} = \frac{\partial \phi}{\partial y} \quad \rightarrow \quad \varphi = \beta\, \phi \tag{1.40}$$

Prandtl–Glauert rule



Using (1.33) it is possible to obtain relation for C_p:

$$C_p = -\frac{\tilde{u}}{V_\infty} = -\frac{1}{V_\infty}\frac{\partial \phi}{\partial x} = -\frac{1}{\beta\, V_\infty}\frac{\partial \varphi}{\partial x} = -\frac{\hat{u}}{\beta\, V_\infty} = \frac{1}{\beta}\,\hat{C}_p \qquad (1.41)$$

Since moment and lift coefficients are integrals of C_p finally we can obtain:

$$C_p = \frac{\hat{C}_p}{\sqrt{1-M_\infty^2}} \qquad C_L = \frac{\hat{C}_L}{\sqrt{1-M_\infty^2}} \qquad C_M = \frac{\hat{C}_M}{\sqrt{1-M_\infty^2}} \qquad (1.42)$$

Other high order rules

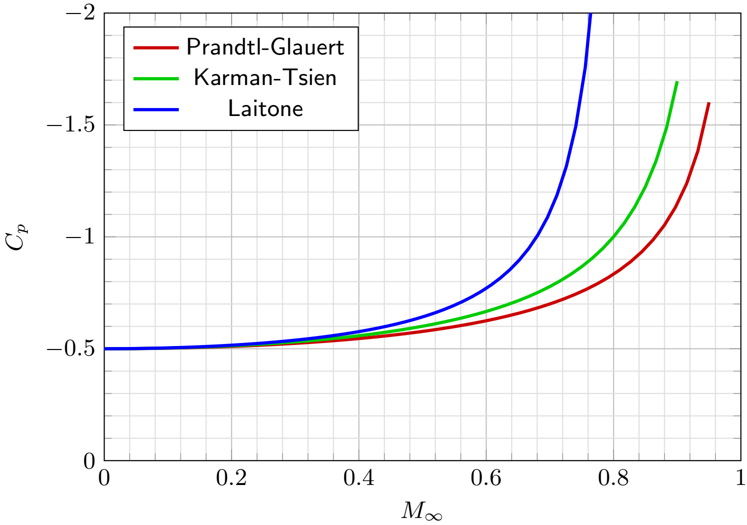
Karman–Tsien rule:

$$C_p = \frac{\hat{C}_p}{\sqrt{1-M_\infty^2} + \frac{M_\infty^2}{1+\sqrt{1-M_\infty^2}} \, \frac{\hat{C}_p}{2}} \tag{1.43}$$

Laitone rule:

$$C_p = \frac{\hat{C}_p}{\sqrt{1-M_\infty^2} + \frac{M_\infty^2 \left(1+\frac{k-1}{2} M_\infty^2\right)}{2\sqrt{1-M_\infty^2}} \, \hat{C}_p} \tag{1.44}$$

Other high order rules



Linearized supersonic flow

In supersonic flow ($M_\infty > 1$) the equation (1.21) can be written:

$$\lambda^2 \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \text{where:} \qquad \lambda = \sqrt{M_\infty^2 - 1} \tag{1.45}$$

This is the wave equation where the wave propagation speed is equal λ. The general solution to this problem has form:

$$\phi = f_+(x + \lambda\, y) + f_-(x - \lambda\, y) \tag{1.46}$$

where f_+ i f_- are arbitrarily chosen functions. The values of f_+ and f_- are constant along lines defined by:

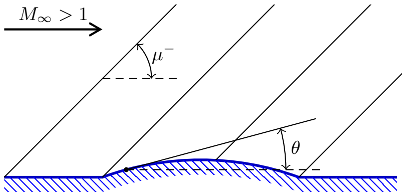
$$x + \lambda\, y = const \qquad x - \lambda\, y = const \tag{1.47}$$

Such lines are called **characteristics** of the wave equation. the angle of inclination of the characteristics can be obtained from (1.46):

$$\frac{dy}{dx} = \pm \frac{1}{\lambda} = \pm \frac{1}{\sqrt{M_\infty^2 - 1}} = \mathrm{tg}(\mu^\pm) \tag{1.48}$$

Characteristics of the equation (1.45) are identical to the Mach lines.

Linearized supersonic flow



The boundary condition for zero normal velocity must be satisfied on the boundary of the domain or airfoil':

$$\mathrm{tg}(\theta) = \frac{\tilde{v}}{V_\infty + \tilde{u}} \approx \frac{\tilde{v}}{V_\infty} \tag{1.49}$$

Using general solution to the wave equation (1.46):

$$\tilde{u} = \frac{\partial \phi}{\partial x} = f'_+ + f'_- \qquad \tilde{v} = \frac{\partial \phi}{\partial y} = \lambda\,(f'_+ - f'_-) \quad \rightarrow$$

$$\rightarrow \quad \tilde{u} = \frac{\tilde{v}}{\lambda}\left[\frac{f'_+ + f'_-}{f'_+ - f'_-}\right] \tag{1.50}$$

Linearized supersonic flow

Using boundary condition (1.49) we can obtain:

$$\tilde{v} = V_\infty \, \mathrm{tg}(\theta) \quad \rightarrow \quad \tilde{u} \approx \frac{V_\infty \, \theta}{\lambda} \left[\frac{f'_+ + f'_-}{f'_+ - f'_-} \right] \tag{1.51}$$

After substituting (1.51) into (1.33):

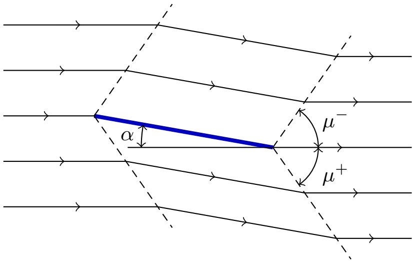
$$C_p = - \frac{2\,\theta}{\sqrt{M_\infty^2 - 1}} \left[\frac{f'_+ + f'_-}{f'_+ - f'_-} \right] \tag{1.52}$$

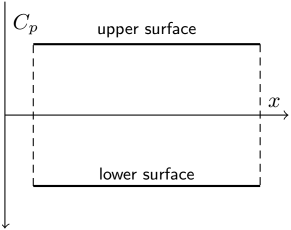
Functions f_+ and f_- can be chosen arbitrarily. In order to simplify the problem we will consider only two cases: f_+ is constant or f_- is constant. Since characteristics are identical to Mach lines their type (sign) should be chosen such that the disturbances will be propagated with the flow.

$$C_p^{U/L} = \pm \frac{2\,\theta}{\sqrt{M_\infty^2 - 1}} \tag{1.53}$$

wher: U – upper airfoil surface, L – lower airfoil surface

Linearized supersonic flow - flat plate





Upper surface – characteristic μ^- and $\theta = -\alpha$:

$$C_p^U = -\frac{2\,\alpha}{\sqrt{M_\infty^2 - 1}} \qquad (1.54)$$

Lower surface – characteristic μ^+ and $\theta = -\alpha$:

$$C_p^L = \frac{2\,\alpha}{\sqrt{M_\infty^2 - 1}} \qquad (1.55)$$

Linearized supersonic flow - flat plate

Force coefficients in normal and tangent directions:

$$C_n = \frac{1}{l_{ac}} \int_0^{l_{ac}} (C_p^L - C_p^U)\, dx = \frac{4\,\alpha}{\sqrt{M_\infty^2 - 1}} \tag{1.56}$$

$$C_t = 0$$

Lift coefficient:

$$C_L = C_n\; cos(\alpha) - C_t\; sin(\alpha) \approx C_n \quad \rightarrow \quad C_L = \frac{4\,\alpha}{\sqrt{M_\infty^2 - 1}} \tag{1.57}$$

Drag coefficient:

$$C_D = C_n\; sin(\alpha) + C_t\; cos(\alpha) \approx C_n\; \alpha \quad \rightarrow \quad C_D = \frac{4\,\alpha^2}{\sqrt{M_\infty^2 - 1}} \tag{1.58}$$

In potential supersonic flow, there exists a nonzero drag force (in contrary to subsonic flow – s'Alembert paradox). This force is called **wave drag**.

Linearized supersonic flow - flat plate

Example:

Flow past a flat plate for $\alpha = 10^\circ$, $M_\infty = 2$ and $p_\infty = 1$

Simplified theory - linearized supersonic flow:

$$C_L = \frac{4\,\alpha}{\sqrt{M_\infty^2 - 1}} = 0.403$$

$$C_D = \frac{4\,\alpha^2}{\sqrt{M_\infty^2 - 1}} = 0.0703$$

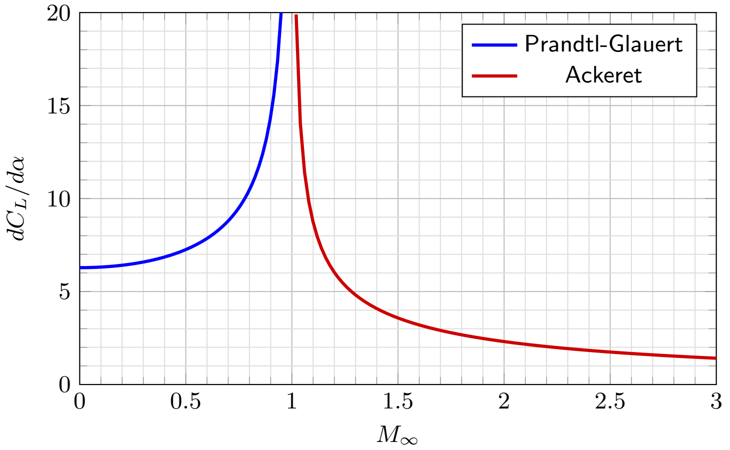
Exact theory based on the oblique shock wave and Prandtl-Meyer expansion:

$$C_L = 0.408$$

$$C_D = 0.0719$$

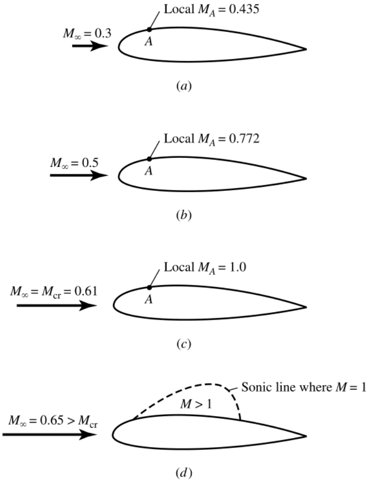
Linearized supersonic flow - $dC_L/d\alpha$

Incompressible flows:	$dC_L/d\alpha = 2\pi$
Subsonic flows (P-G rule):	$dC_L/d\alpha = 2\pi/\sqrt{M_\infty^2 - 1}$
Supersonic flows:	$dC_L/d\alpha = 4/\sqrt{M_\infty^2 - 1}$



Transonic flow past a 2D airfoil

Critical Mach number



Critical Mach number M_{cr} is a Mach number of undisturbed flow M_∞ for which maximum of the local velocity at some point on the airfoil surface is equal to the speed of sound.

If $M_\infty < M_{cr}$ then flow is fully subsonic (a i b)

If $M_\infty > M_{cr}$ then there exists a supersonic flow region (d)

Critical Mach number

For izentropic flow:

$$\frac{p}{p_\infty} = \frac{p}{p_0}\,\frac{p_0}{p_\infty} = \left(\frac{1+\frac{k-1}{2}M_\infty^2}{1+\frac{k-1}{2}M^2}\right)^{\frac{k}{k-1}} \tag{2.1}$$

Using (1.24):

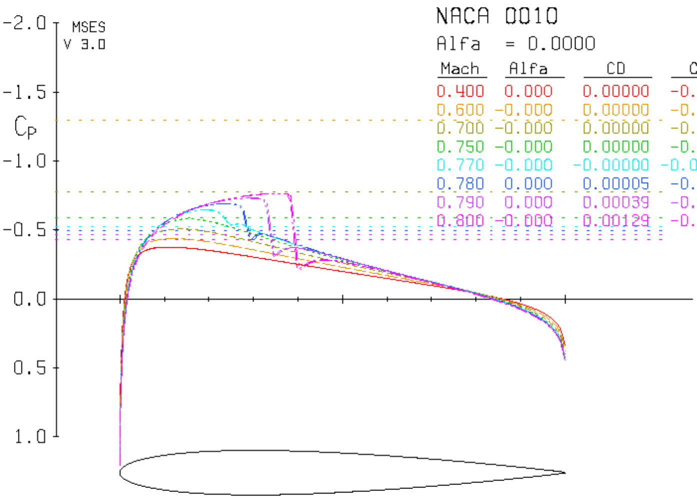
$$C_p = \frac{2}{k\,M_\infty^2}\left[\left(\frac{1+\frac{k-1}{2}M_\infty^2}{1+\frac{k-1}{2}M^2}\right)^{\frac{k}{k-1}} - 1\right] \tag{2.2}$$

where M is local Mach number at some point on airfoil surface.
If $M = 1$ then:

$$C_p^* = \frac{2}{k\,M_\infty^2}\left[\left(\frac{1+\frac{k-1}{2}M_\infty^2}{1+\frac{k-1}{2}}\right)^{\frac{k}{k-1}} - 1\right] \tag{2.3}$$

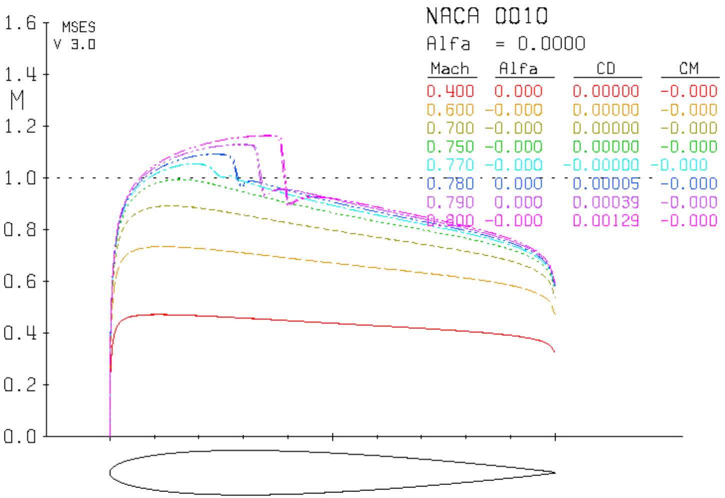
C_p^* is a critical pressure coefficient. If at some point on airfoil surface $C_p > C_p^*$ then at this point $M > 1$, otherwise if $C_p < C_p^*$ then $M < 1$.

Critical pressure coefficient

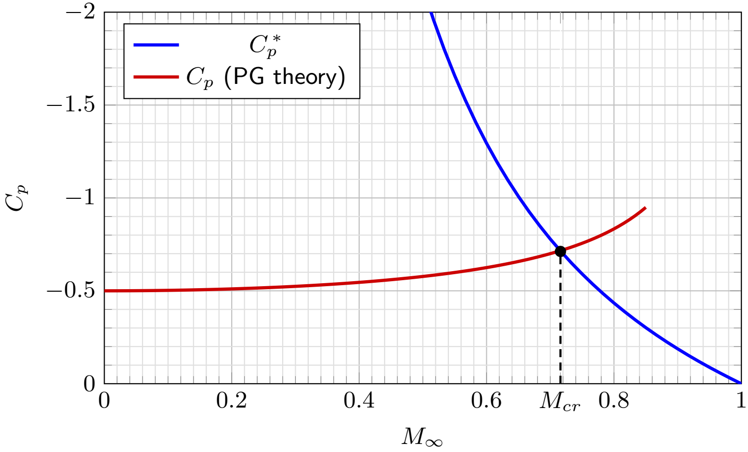


Mach	Alfa	CD	CM
0.400	0.000	0.00000	-0.000
0.600	-0.000	0.00000	-0.000
0.700	-0.000	0.00000	-0.000
0.750	-0.000	0.00000	-0.000
0.770	-0.000	-0.00000	-0.000
0.780	0.000	0.00005	-0.000
0.790	0.000	0.00039	-0.000
0.800	-0.000	0.00129	-0.000

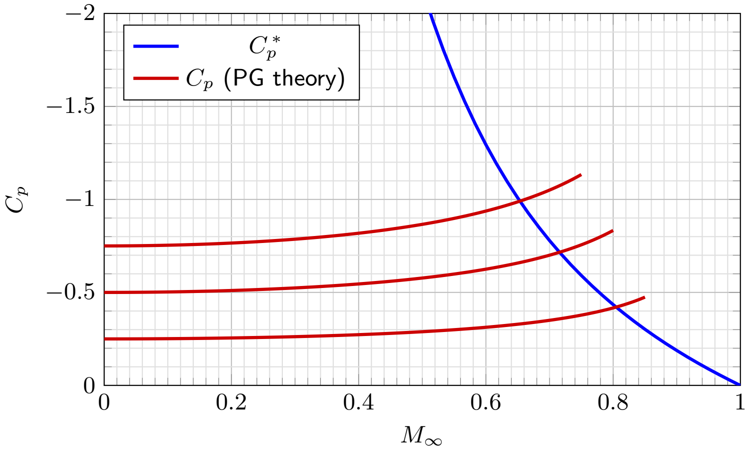
Critical pressure coefficient



Krytyczny współczynnik ciśnienia



Critical pressure coefficient



Critical pressure coefficient – airfoil thickness