Warsaw University of Technology

Faculty of Power and Aeronautical Engineering

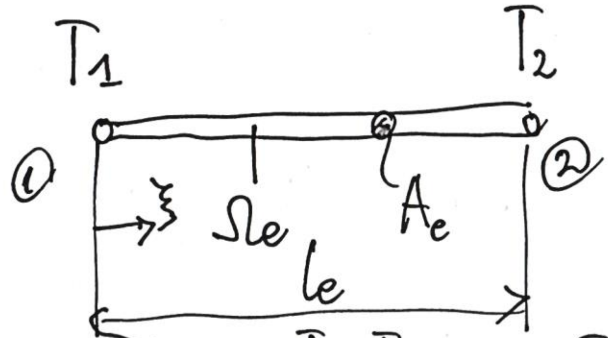
WARSAW UNIVERSITY OF TECHNOLOGY

Institute of Aeronautics and Applied Mechanics

Finite element method 2 (FEM 2)

Heat transfer - examples

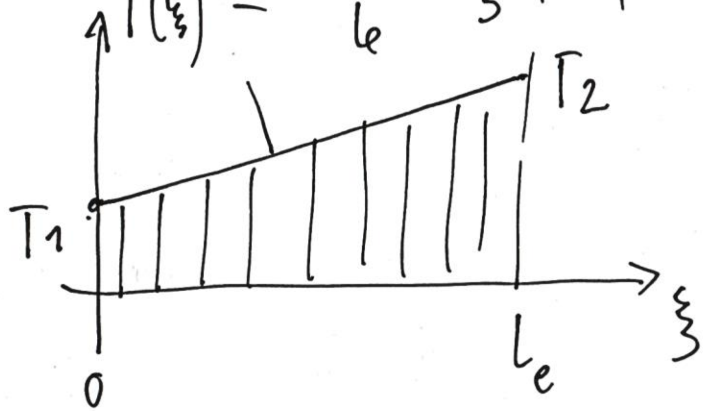
EXAMPLE. CONDUCTIVITY MATRIX OF A BAR ELEMENT



(1D)

$$T(\xi) = \frac{T_2 - T_1}{l_e}\cdot \xi + T_1 = \underbrace{\left(1 - \frac{\xi}{l_e}\right)}_{N_1(\xi)}\cdot T_1 + \underbrace{\left(\frac{\xi}{l_e}\right)}_{N_2(\xi)}\cdot T_2 =$$

$$= \lfloor N_1, N_2 \rfloor \cdot \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}_e$$



$$d\Omega_e = A_e \cdot d\xi$$

$$h_{ij}^{e} = \int_{\Omega_e} \lambda \frac{\partial N_i}{\partial \xi}\cdot \frac{\partial N_j}{\partial \xi}\, d\Omega_e =$$

$$= \int_0^{l_e} A_e \cdot \lambda \cdot \frac{\partial N_i}{\partial \xi}\cdot \frac{\partial N_j}{\partial \xi}\, d\xi$$

$$\frac{\partial N_i}{\partial \xi} = \frac{\partial}{\partial T_i}\left(\frac{\partial T}{\partial \xi}\right) = \frac{\partial}{\partial T_i}\left(\left\lfloor \frac{\partial N_1}{\partial \xi}, \frac{\partial N_2}{\partial \xi} \right\rfloor \cdot \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}_e\right)$$

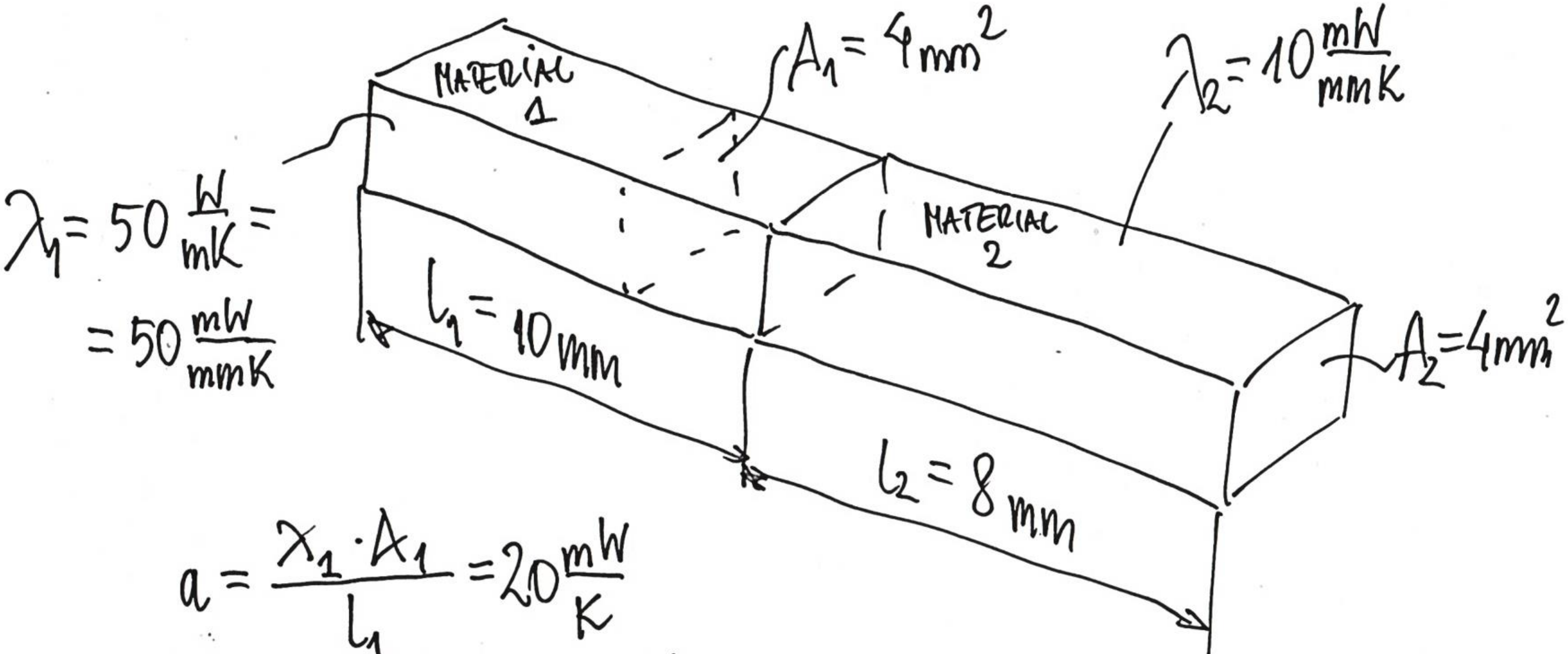
$$\frac{\partial N_1}{\partial \xi} = \frac{\partial}{\partial T_1}\left\lfloor -\frac{1}{l_e}, \frac{1}{l_e} \right\rfloor \cdot \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}_e = \frac{\partial}{\partial T_1}\left(-\frac{T_1}{l_e} + \frac{T_2}{l_e}\right) = -\frac{1}{l_e}$$

$$= -\frac{1}{l_e}, \quad \frac{\partial N_2}{\partial \xi} = \frac{\partial}{\partial T_2}\left(\left\lfloor -\frac{1}{l_e}, \frac{1}{l_e} \right\rfloor \cdot \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}_e\right) = \frac{1}{l_e}$$

$$h_{11}^e = \int_0^{l_e} A_e \cdot \lambda \cdot \left(-\frac{1}{l_e}\right) \cdot \left(-\frac{1}{l_e}\right) d\xi = \frac{A_e \lambda}{l_e^2} \cdot \xi \Big|_0^{l_e} = \frac{A_e \lambda}{l_e}$$

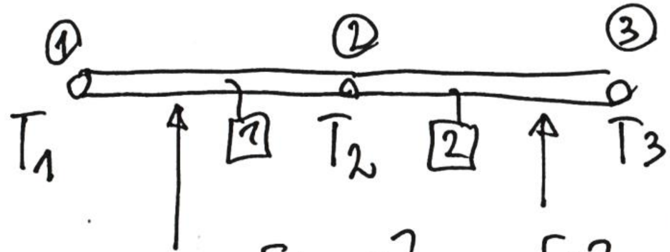
$$\underset{2\times 2}{\left[h\right]_e} = \begin{bmatrix} \frac{A_e\lambda}{l_e} & -\frac{A_e\lambda}{l_e} \\ -\frac{A_e\lambda}{l_e} & \frac{A_e\lambda}{l_e} \end{bmatrix} = \frac{A_e\lambda}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

EXAMPLE. FIND TEMPERATURE DISTRIBUTION IN A BAR
THERMAL GRADIENTS AND FLUXES, AND THE HEAT RATE



$$a = \frac{\lambda_1 \cdot A_1}{l_1} = 20 \frac{mW}{K}$$

$$b = \frac{\lambda_2 \cdot A_2}{l_2} = 5 \frac{mW}{K}$$



$$\{T\}_{3\times 1} = \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix}$$

$$\Delta T = T - T_0$$
$$T_0 = 0°C \Rightarrow \Delta T = T$$

$$[h]_1 = \begin{bmatrix} a & -a \\ -a & a \end{bmatrix} \qquad [h]_2 = \begin{bmatrix} b & -b \\ -b & b \end{bmatrix}$$

$$\underset{3\times 3}{[H]} = \begin{bmatrix} a & -a & 0 \\ -a & a+b & -b \\ 0 & -b & b \end{bmatrix} \quad ; \quad [H]\cdot\{T\} + \{F\} = \{0\}$$
$$[H]\cdot\{T\} = -\{F\}$$

CASE 1°)

$T_1 = T_L = 100K$ $\quad$ $T_2 = ?$ $\quad$ $T_3 = T_R = 20K$

$$[H]\cdot\{T\} = -\begin{Bmatrix} F_1 \\ 0 \\ F_3 \end{Bmatrix}$$

$$[H]\cdot\begin{Bmatrix} 0 \\ T_2 \\ 0 \end{Bmatrix} + [H]\cdot\begin{Bmatrix} T_L \\ 0 \\ T_R \end{Bmatrix} = -\begin{Bmatrix} F_1 \\ 0 \\ F_3 \end{Bmatrix}$$

$$\underset{3\times 3}{\left[H\right]} \cdot \begin{Bmatrix} 0 \\ T_2 \\ 0 \end{Bmatrix} = -\begin{Bmatrix} F_1 \\ 0 \\ F_3 \end{Bmatrix} - \left[H\right] \cdot \begin{Bmatrix} T_L \\ 0 \\ T_R \end{Bmatrix} \quad ; \quad \left[H\right] = \begin{bmatrix} a & -a & 0 \\ -a & a+b & -b \\ 0 & -b & b \end{bmatrix}$$

$$\left[H\right] \cdot \begin{Bmatrix} 0 \\ T_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -F_1 \\ 0 \\ -F_3 \end{Bmatrix} - \begin{Bmatrix} a \cdot T_L \\ -a \cdot T_L - b \cdot T_R \\ b \cdot T_R \end{Bmatrix} = \begin{Bmatrix} -F_1 - a \cdot T_L \\ a\, T_L + b \cdot T_R \\ -F_3 - b \cdot T_R \end{Bmatrix}$$

$$\begin{bmatrix} \, & \, & \, \\ \, & a+b & \, \\ \, & \, & \, \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ T_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -F_1 - a \cdot T_L \\ a\, T_L + b \cdot T_R \\ -F_3 - b \cdot T_R \end{Bmatrix}$$

$$(a+b) \cdot T_2 = a\, T_L + b \cdot T_R$$

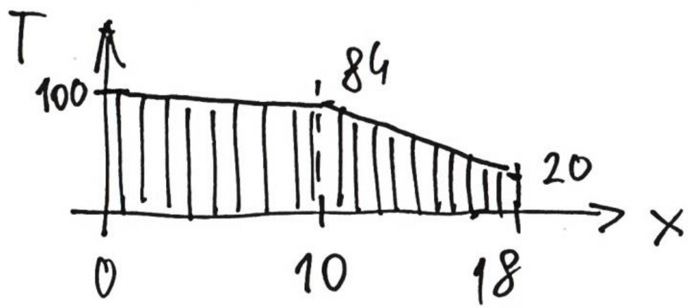
$$T_2 = \frac{a \cdot T_L + b \cdot T_R}{(a+b)} = 84\,K$$

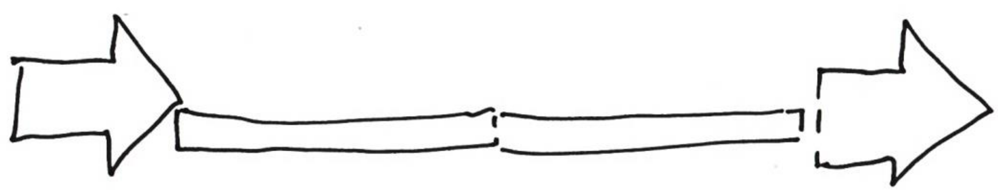
$$\begin{bmatrix} a & -a & 0 \\ -a & a+b & -b \\ 0 & -b & b \end{bmatrix} \cdot \begin{Bmatrix} T_L \\ T_2 \\ T_R \end{Bmatrix} = \begin{Bmatrix} -F_1 \\ 0 \\ -F_3 \end{Bmatrix}$$

$$a \cdot T_L - a \cdot T_2 = -F_1 \implies F_1 = \frac{-ab(T_L - T_R)}{a+b} = -320\,mW$$

$$0 \cdot T_L - b \cdot T_2 + b \cdot T_R = -F_3 \implies$$

$$F_3 = \frac{a \cdot b \cdot (T_L - T_R)}{a + b} = 320\,\text{mW}$$





heat rate

$-F_1 = +320\,\text{mW}$

(heat flows into the system)

$-F_3 = -320\,\text{mW}$

(heat flows out of the system)

ELEMENT SOLUTION:

thermal gradient

$$\frac{\partial T}{\partial \xi} = \frac{\partial \lfloor N_1, N_2 \rfloor}{\partial \xi} \cdot \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}_e = -\frac{1}{l_e} \cdot T_1 + \frac{1}{l_e} \cdot T_2 = \frac{(T_2 - T_1)_e}{l_e}$$

$\boxed{1}$: $$\left.\frac{\partial T}{\partial \xi}\right|_1 = \frac{T_2 - T_L}{L_1} = -1.6 \frac{K}{mm}$$

$\boxed{2}$ $$\left.\frac{\partial T}{\partial \xi}\right|_2 = \frac{T_R - T_2}{l_2} = -8 \frac{K}{mm}$$

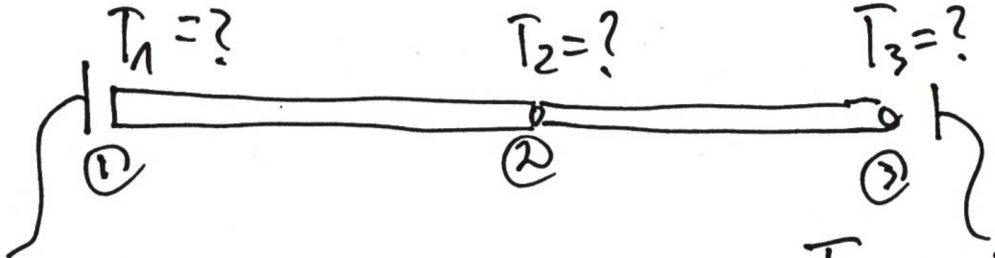
thermal flux:

$$q_e = -\lambda_e \cdot \frac{\partial T}{\partial \xi}.$$

$\boxed{1}$ $$q_1 = -\lambda_1 \cdot \left.\frac{\partial T}{\partial \xi}\right|_1 = -50 \frac{mW}{mmK} \cdot (-1.6) \cdot \frac{K}{mm} = 80 \frac{mW}{mm^2}$$

$\boxed{2}$ $$q_2 = -\lambda_2 \cdot \left.\frac{\partial T}{\partial \xi}\right|_2 = -10 \frac{mW}{mmK} \cdot (-8) \frac{K}{mm} = 80 \frac{mW}{mm^2}$$

CASE 2



$T_{L\infty} = 200K$

$\alpha_L = 1 \frac{mW}{mm^2 K}$

$T_{R\infty} = 50K$

$\alpha_R = 0.2 \frac{mW}{mm^2 K} \left(= 200 \frac{W}{m^2 K}\right)$

FILM COEFFICIENT

$$q_L = \alpha_L \left(T_1 - T_{L\infty}\right)$$

$$q_R = \alpha_R \left(T_3 - T_{R\infty}\right)$$

$Q_L = q_L \cdot A_1$, $Q_R = q_R \cdot A_2$ heat rate. (mW)

$$[H]\cdot\{T\}+\{F\}=\{0\}$$

$$[H]\cdot\{T\}+\begin{Bmatrix} Q_L \\ 0 \\ Q_R \end{Bmatrix}=\{0\}$$

$$[H]\cdot\{T\}=-\begin{Bmatrix} Q_L \\ 0 \\ Q_R \end{Bmatrix}=\begin{Bmatrix} \alpha_L A_1 (T_{L\infty}-T_1) \\ 0 \\ \alpha_R A_2 (T_{R\infty}-T_3) \end{Bmatrix}$$

$$[H]\cdot\{T\}+\begin{Bmatrix} \alpha_L A_1 T_1 \\ 0 \\ \alpha_R A_2 T_3 \end{Bmatrix}=\begin{Bmatrix} \alpha_L A_1 T_{L\infty} \\ 0 \\ \alpha_R A_2 T_{R\infty} \end{Bmatrix}$$

$$[H]\cdot\{T\} + \begin{Bmatrix} \alpha_L A_1 T_1 + 0\cdot T_2 + 0\cdot T_3 \\ 0\cdot T_1 + 0\cdot T_2 + 0\cdot T_3 \\ 0\cdot T_1 + 0\cdot T_2 + \alpha_R\cdot A_2\cdot T_3 \end{Bmatrix} = \begin{Bmatrix} \alpha_L A_1 T_{L\infty} \\ 0 \\ \alpha_R A_2 T_{R\infty} \end{Bmatrix}$$

$$\Big(\underbrace{[H]}_{\text{conductive part}} + [\alpha A]\Big)\cdot\{T\} = \begin{Bmatrix} \alpha_L A_1 T_{L\infty} \\ 0 \\ \alpha_R A_2 T_{R\infty} \end{Bmatrix}$$

$$\underbrace{[\alpha A]}_{\text{convective part}}{}_{3\times3} = \begin{bmatrix} \alpha_L A_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha_R A_2 \end{bmatrix}$$

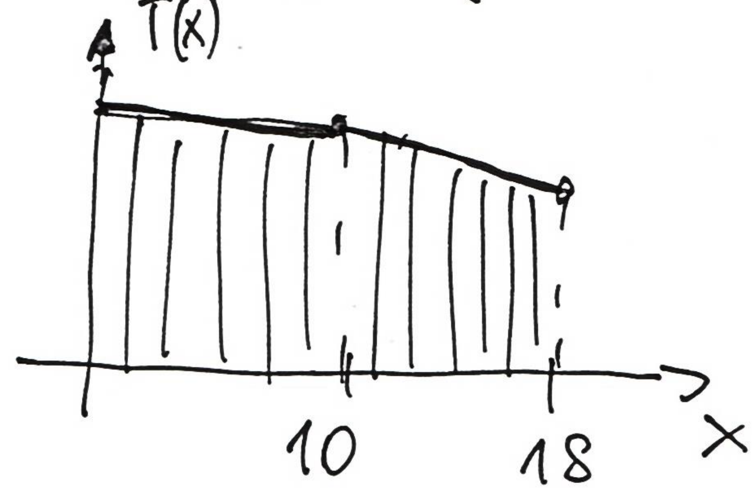
$$\begin{bmatrix} a+\alpha_L A_1 & -a & 0 \\ -a & a+b & -b \\ 0 & -b & b+\alpha_R A_2 \end{bmatrix}\cdot\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} \alpha_L A_1 T_{L\infty} \\ 0 \\ \alpha_R A_2 T_{R\infty} \end{Bmatrix}$$

$$\{T\} = \big([H]+[\alpha A]\big)^{-1}\cdot\begin{Bmatrix} \alpha_L A_1 T_{L\infty} \\ 0 \\ \alpha_R A_2 T_{R\infty} \end{Bmatrix}$$

$$\left([H]+[\alpha A]\right)=\begin{bmatrix}24 & -20 & 0\\ -20 & 25 & -5\\ 0 & -5 & 5.8\end{bmatrix}$$

$$\left([H]+[\alpha A]\right)^{-1}=\begin{bmatrix}0.2143 & 0.2071 & 0.1786\\ \leftarrow \text{sym} & 0.2486 & 0.2143\\ & \rightarrow & 0.3571\end{bmatrix}$$

$$\{T\}=\begin{Bmatrix}178.57\\ 174.29\\ 157.14\end{Bmatrix}$$



element solution:

$\boxed{1}$ $$\left.\frac{\partial T}{\partial \xi}\right|_1 = \frac{T_2 - T_1}{l_1} = -0.4286 \,\frac{K}{mm}$$

$$q_1 = -\lambda_1 \cdot \left.\frac{\partial T}{\partial \xi}\right|_1 = 21.43 \,\frac{mW}{mm^2}$$

$\boxed{2}$ $$\left.\frac{\partial T}{\partial \xi}\right|_2 = \frac{T_3 - T_2}{l_2} = -2.1429 \,\frac{K}{mm}$$

$$q_2 = -\lambda_2 \cdot \left.\frac{\partial T}{\partial \xi}\right|_2 = 21.43 \,\frac{mW}{mm^2}$$

$$q_L = \alpha_L \left(T_1 - T_{L\infty}\right) = -21.43 \,\frac{mW}{mm^2}$$

$$q_R = \alpha_R \cdot \left(T_3 - T_{R\infty}\right) = 21.43 \,\frac{mW}{mm^2}$$

$-Q_L = -q_L \cdot A_1 = 85.72\,mW$ $\qquad$ $-Q_R = q_R \cdot A_2 = -85.72\,mW$

heat rate