Warsaw University of Technology

Faculty of Power and Aeronautical Engineering

WARSAW UNIVERSITY OF TECHNOLOGY

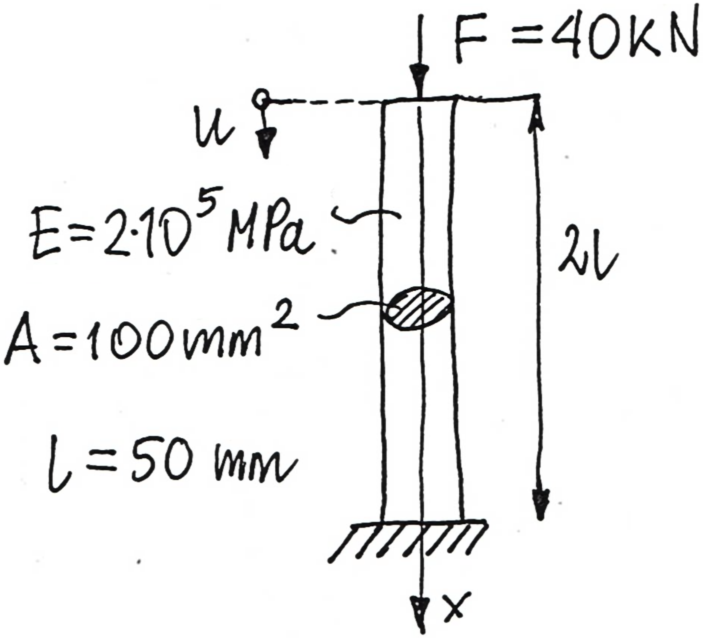
Institute of Aeronautics and Applied Mechanics

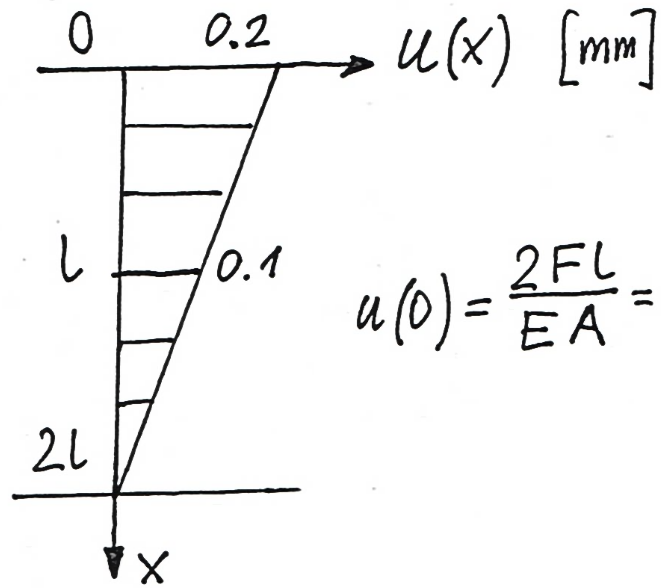
Finite element method 2 (FEM 2)

Contact problem

CONTACT PROBLEM

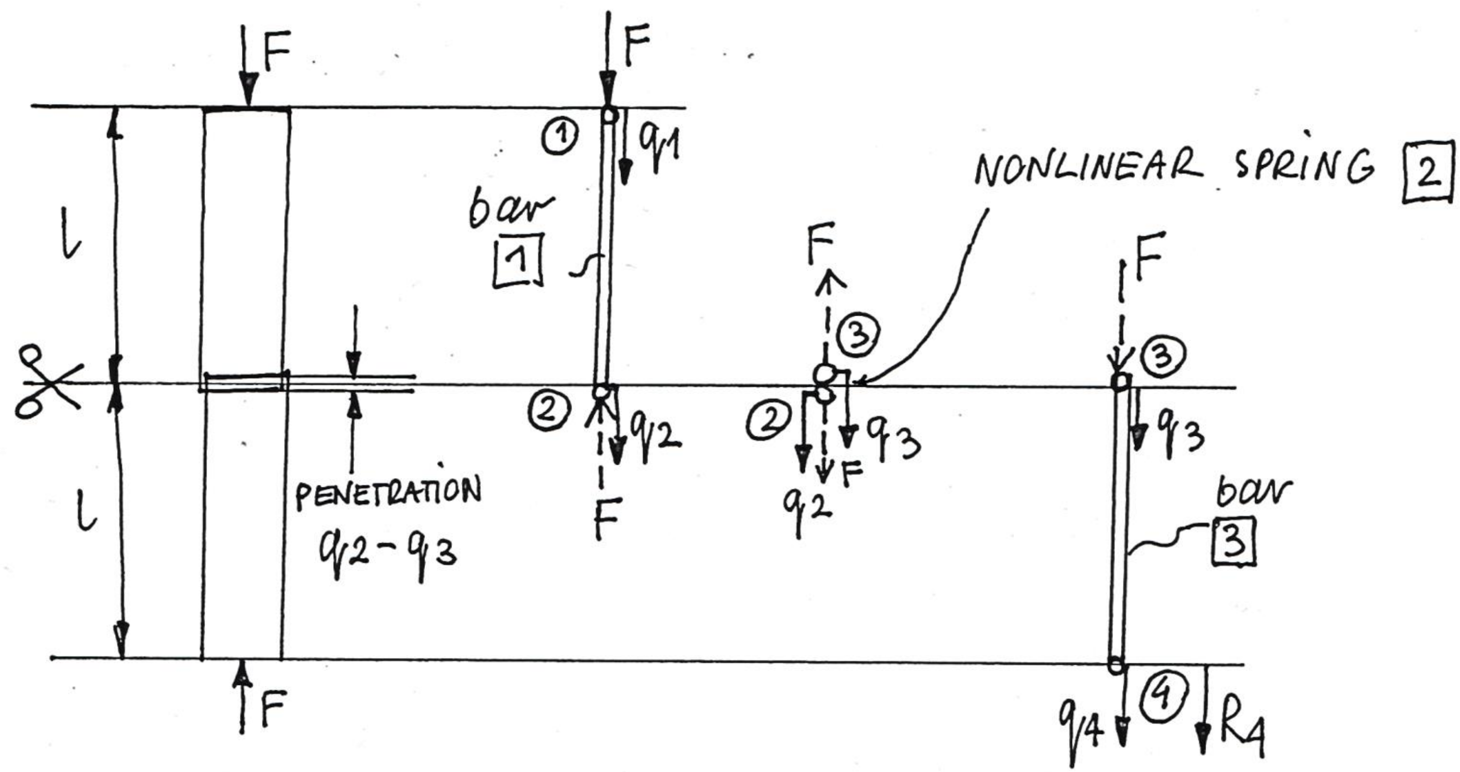
1°) LINEAR SOLUTION OF A COLUMN





$$u(0) = \frac{2Fl}{EA} = 0.2\,mm$$

2°) NONLINEAR SOLUTION (COLUMN DIVIDED INTO TWO PARTS)

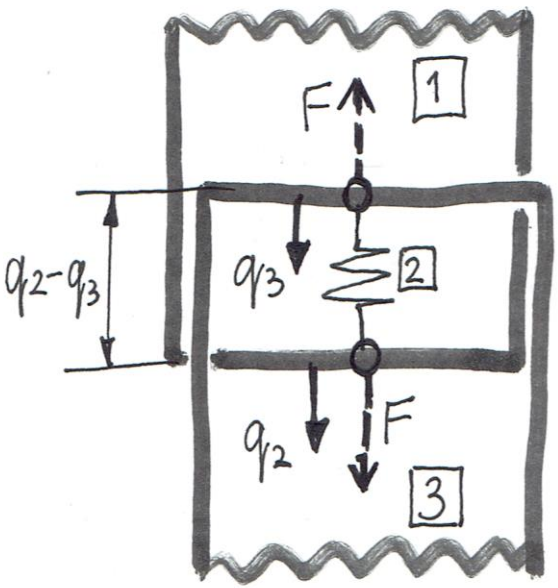


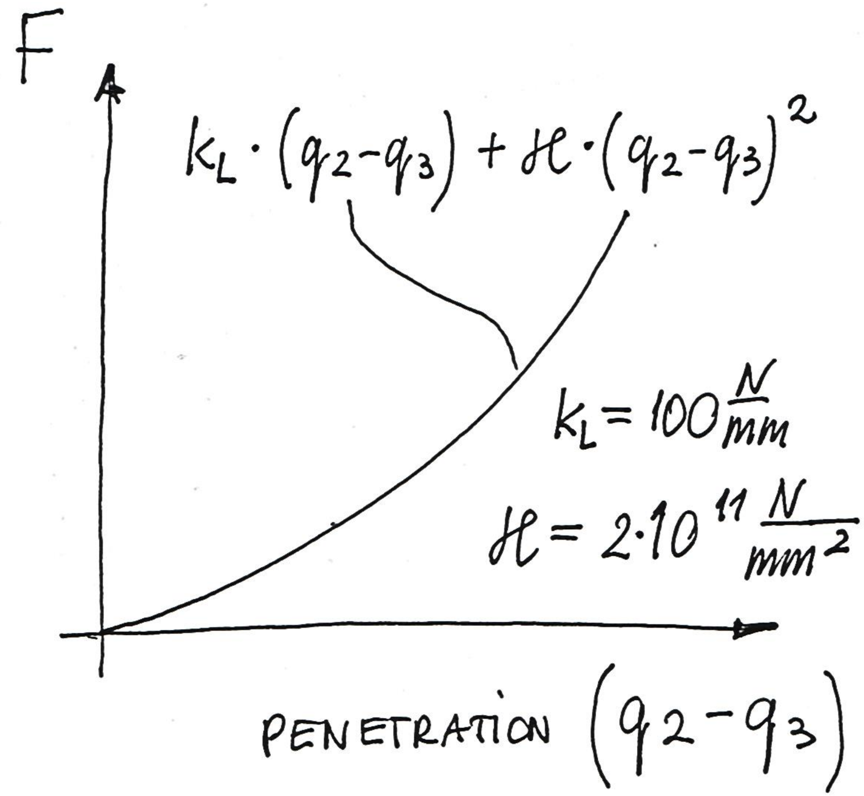
A SPRING IS ADDED TO STOP THE PENETRATION.

THE CONTACT AREA IS USUALLY A NONLINEAR FUNCTION OF LOAD. IN THE EXAMPLE THE AREA IS CONSTANT, SO INSTEAD WE ASSUME THAT THE SPRING HAS A NONLINEAR CHARACTERISTICS.

FORCE IN THE SPRING :

$$F = k \cdot (q_2 - q_3) \quad ; \text{ where } \quad k = k_L + \mathcal{H}\,(q_2 - q_3)$$





global vector of nodal parameters:

$$\lfloor q \rfloor_{1\times 4} = \lfloor q_1, q_2, q_3, q_4 \rfloor$$

global load vector:

$$\lfloor F \rfloor_{1\times 4} = \lfloor F, 0, 0, R_4 \rfloor$$

local stiffness matrices:

$$\underset{2\times 2}{[K]_1} = \underset{2\times 2}{[K]_3} = \begin{bmatrix} a & -a \\ -a & a \end{bmatrix} \quad ; \quad a = \frac{EA}{l} = 4 \cdot 10^5 \frac{N}{mm}$$

$$\underset{2\times 2}{[K(q)]_2} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \quad ; \quad k = k_L + \varkappa (q_2 - q_3)$$

global stiffness matrix :

$$\underset{4\times4}{\left[K(q)\right]} = \begin{bmatrix} [k]_1 & & 0 & 0 \\ & [k]_2 & & 0 \\ 0 & & & \\ 0 & 0 & & [k]_3 \end{bmatrix} = \begin{bmatrix} a & -a & 0 & 0 \\ -a & a+k & -k & 0 \\ 0 & -k & a+k & -a \\ 0 & 0 & -a & a \end{bmatrix}$$

set of nonlinear equations :

$$\underset{4\times4}{\left[K(q)\right]} \cdot \underset{4\times1}{\{q\}} = \underset{4\times1}{\{F\}}$$

boundary condition $q_4 = 0 \Rightarrow N = NDOF - NOF = 4 - 1 = 3$

$$\underset{3\times3}{\left[K(q)\right]} \cdot \underset{3\times1}{\{q\}} = \underset{3\times1}{\{F\}}$$

$$\begin{bmatrix} a & -a & 0 \\ -a & a+k & -k \\ 0 & -k & a+k \end{bmatrix} \cdot \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \\ 0 \end{Bmatrix} \; ; \quad k = f(q_2, q_3)$$

(Newton – Raphson method)

tangent matrix :

$$\underset{3\times 3}{[K_T]} = \frac{\partial \underset{3\times 1}{\{F\}}}{\partial \underset{1\times 3}{\lfloor q \rfloor}} = \left[\frac{\partial \underset{3\times 1}{\{F\}}}{\partial q_1} , \frac{\partial \underset{3\times 1}{\{F\}}}{\partial q_2} , \frac{\partial \underset{3\times 1}{\{F\}}}{\partial q_3} \right]$$

$$\frac{\partial \underset{3\times 1}{\{F\}}}{\partial q_1} = \frac{\partial\left(\underset{3\times 3}{[K(q)]}\cdot \underset{3\times 1}{\{q\}}\right)}{\partial q_1} = \underbrace{\frac{\partial \underset{3\times 3}{[K(q)]}}{\partial q_1}}_{\substack{\parallel \\ \underset{3\times 3}{[0]}}} \cdot \underset{3\times 1}{\{q\}} + \underset{3\times 3}{[K(q)]}\cdot \frac{\partial \underset{3\times 1}{\{q\}}}{\partial q_1} = \underset{3\times 3}{[K(q)]}\cdot \begin{Bmatrix} \frac{\partial q_1}{\partial q_1} \\ \frac{\partial q_2}{\partial q_1} \\ \frac{\partial q_3}{\partial q_1} \end{Bmatrix} =$$

$$= \begin{bmatrix} a & -a & 0 \\ -a & a+k & -k \\ 0 & -k & a+k \end{bmatrix} \cdot \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} a \\ -a \\ 0 \end{Bmatrix} \quad \text{1st row of } [K]$$

$$\frac{\partial \underset{3\times 1}{\{F\}}}{\partial q_2} = \frac{\partial \left(\underset{3\times 3}{[K(q)]} \cdot \underset{3\times 1}{\{q\}} \right)}{\partial q_2} = \frac{\partial \underset{3\times 3}{[K(q)]}}{\partial q_2} \cdot \underset{3\times 1}{\{q\}} + \underset{3\times 3}{[K(q)]} \cdot \frac{\partial \underset{3\times 1}{\{q\}}}{\partial q_2} =$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\partial k}{\partial q_2} & -\frac{\partial k}{\partial q_2} \\ 0 & -\frac{\partial k}{\partial q_2} & \frac{\partial k}{\partial q_2} \end{bmatrix} \cdot \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} + \underset{3\times 3}{[K(q)]} \cdot \begin{Bmatrix} \frac{\partial q_1}{\partial q_2} \\ \frac{\partial q_2}{\partial q_2} \\ \frac{\partial q_3}{\partial q_2} \end{Bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathcal{H} & -\mathcal{H} \\ 0 & -\mathcal{H} & \mathcal{H} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} + \begin{bmatrix} a & -a & 0 \\ -a & a+k & -k \\ 0 & -k & a+k \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} =$$

$$= \begin{Bmatrix} 0 \\ \mathcal{H}(q_2-q_3) \\ -\mathcal{H}(q_2-q_3) \end{Bmatrix} + \begin{Bmatrix} -a \\ a+k \\ -k \end{Bmatrix} = \begin{Bmatrix} -a \\ a+k_L+2\mathcal{H}(q_2-q_3) \\ -(k_L+2\mathcal{H}(q_2-q_3)) \end{Bmatrix}$$

2nd row of [K]

$$\frac{\partial \underset{3\times1}{\{F\}}}{\partial q_3} = \frac{\partial\left(\underset{3\times3}{[K(q)]}\cdot\underset{3\times1}{\{q\}}\right)}{\partial q_3} = \frac{\partial \underset{3\times3}{[K(q)]}}{\partial q_3}\cdot\underset{3\times1}{\{q\}} + \underset{3\times3}{[K(q)]}\cdot\frac{\partial \underset{3\times1}{\{q\}}}{\partial q_3} =$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\partial k}{\partial q_3} & -\frac{\partial k}{\partial q_3} \\ 0 & -\frac{\partial k}{\partial q_3} & \frac{\partial k}{\partial q_3} \end{bmatrix}\cdot\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} + \underset{3\times3}{[K(q)]}\cdot\begin{Bmatrix} \frac{\partial q_1}{\partial q_3} \\ \frac{\partial q_2}{\partial q_3} \\ \frac{\partial q_3}{\partial q_3} \end{Bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\mathcal{H} & \mathcal{H} \\ 0 & \mathcal{H} & -\mathcal{H} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} + \begin{bmatrix} a & -a & 0 \\ -a & a+k & -k \\ 0 & -k & a+k \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} =$$

$$= \begin{Bmatrix} 0 \\ -\mathcal{H}(q_2-q_3) \\ \mathcal{H}(q_2-q_3) \end{Bmatrix} + \underbrace{\begin{Bmatrix} 0 \\ -k \\ a+k \end{Bmatrix}}_{\text{3rd row of } [K]} = \begin{Bmatrix} 0 \\ -(k_L + 2\mathcal{H}(q_2-q_3)) \\ a + k_L + 2\mathcal{H}(q_2-q_3) \end{Bmatrix}$$

Thus:

$$\underset{3\times 3}{[K_T]} = \begin{bmatrix} a & -a & 0 \\ -a & a+k_L+2\varkappa(q_2-q_3) & -(k_L+2\varkappa(q_2-q_3)) \\ 0 & -(k_L+2\varkappa(q_2-q_3)) & a+k_L+2\varkappa(q_2-q_3) \end{bmatrix}$$

initial solution:

$$\underset{3\times 1}{\{q\}}_0 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad ; \quad \underset{3\times 3}{[K(q)]}_0 = \begin{bmatrix} a & -a & 0 \\ -a & a+k_L & -k_L \\ 0 & -k_L & a+k_L \end{bmatrix}$$

$$\underset{3\times 3}{[K_T]}_1 = \frac{\partial \underset{N\times 1}{\{F\}}}{\partial \underset{1\times 3}{\lfloor q \rfloor}_0} = \underset{3\times 3}{[K(q)]}_0$$

SOLUTION AT ITERATION „ i ":

$$\underset{N\times N}{[K_T]}_i = \frac{\partial \underset{N\times 1}{\{F\}}}{\partial \underset{1\times N}{\lfloor q \rfloor}_{i-1}} = \begin{bmatrix} a & -a & 0 \\ -a & a+k_L+2\varkappa(q_2-q_3) & -(k_L+2\varkappa(q_2-q_3)) \\ 0 & -(k_L+2\varkappa(q_2-q_3)) & a+k_L+2\varkappa(q_2-q_3) \end{bmatrix}_{i-1}$$

$$\underset{3\times 1}{\{R\}}_i = \underset{3\times 1}{\{F\}} - \underset{3\times 3}{[K(q)]}_{i-1} \cdot \underset{3\times 1}{\{q\}}_{i-1}$$

$$\underset{3\times 1}{\{\Delta q\}}_i = [K_T]_{i-1}^{-1} \cdot \underset{3\times 1}{\{R\}}_i \qquad (i = 1, \dots, n)$$

$$\underset{3\times 1}{\{q\}}_i = \underset{3\times 1}{\{q\}}_{i-1} + \underset{3\times 1}{\{\Delta q\}}_i$$

convergence criteria :

displacement criterion : $$U2_{NORM_i} = \frac{\| \underset{3\times 1}{\{\Delta q\}_i} \|_2}{\| \underset{3\times 1}{\{q\}_i} \|_2} \leq \varepsilon$$

$\varepsilon = 0.0005$

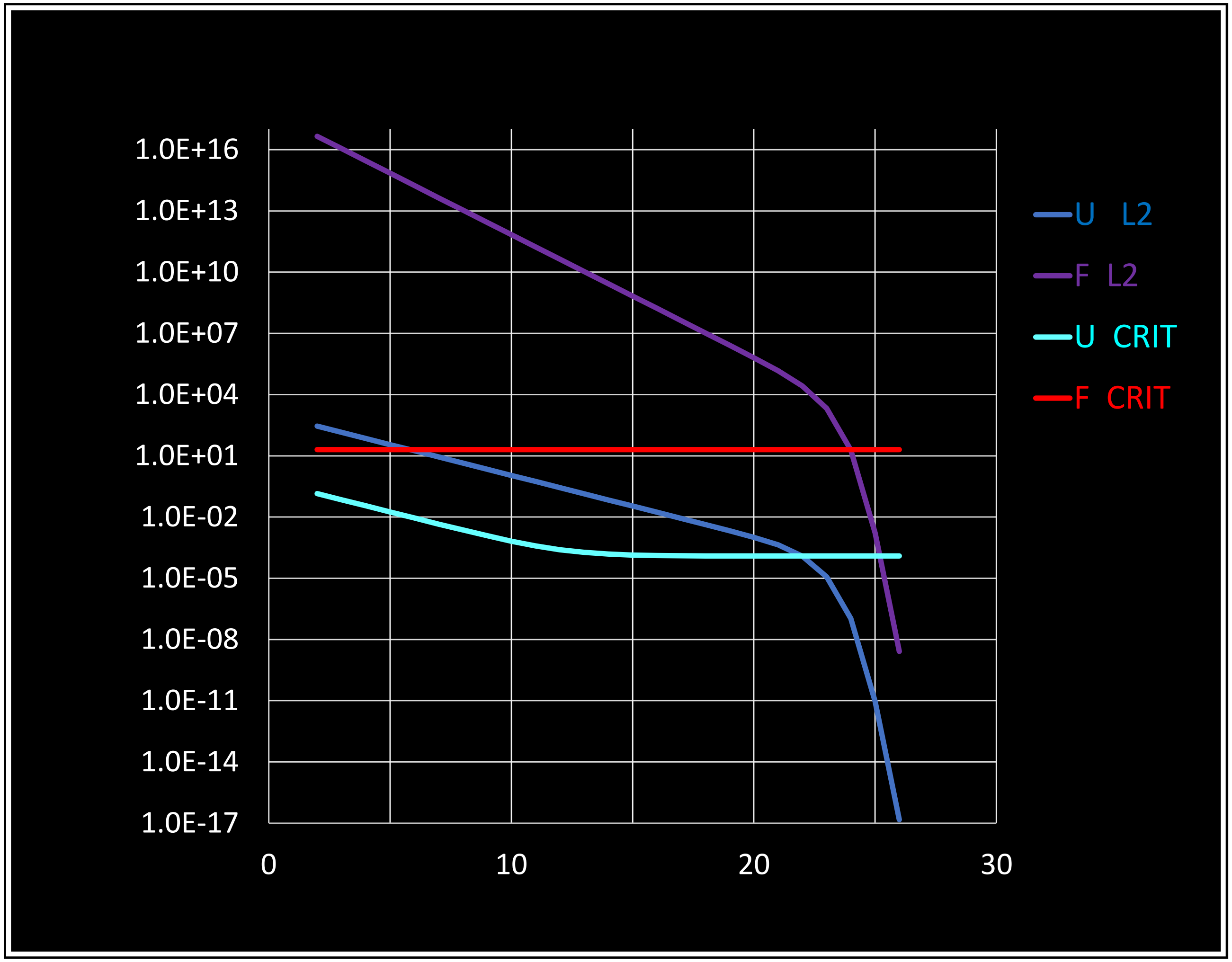
force criterion : $$F2_{NORM_i} = \frac{\| \underset{3\times 1}{\{R\}_i} \|_2}{\| \underset{3\times 1}{\{F\}} \|_2} \leq \delta$$

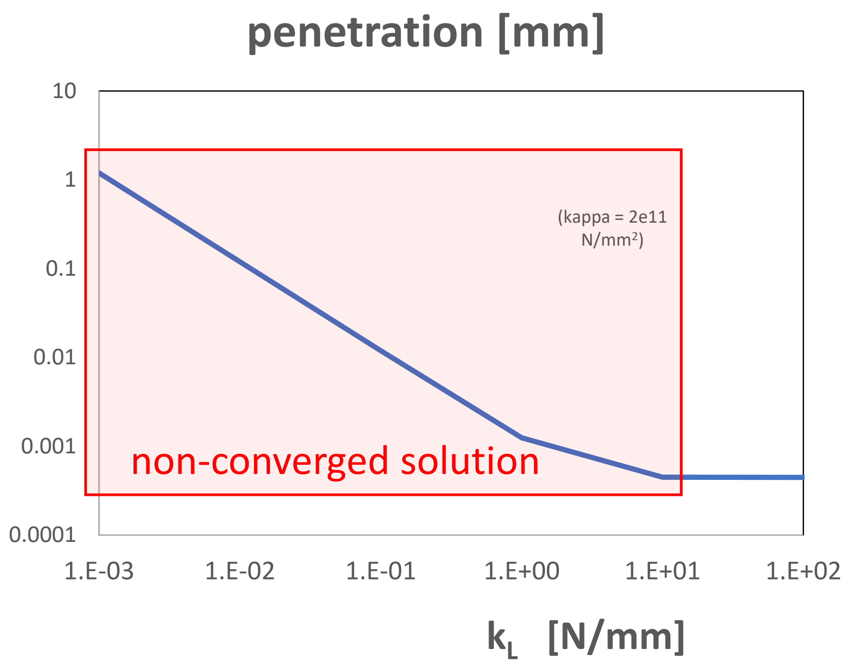
$\delta = 0.0005$

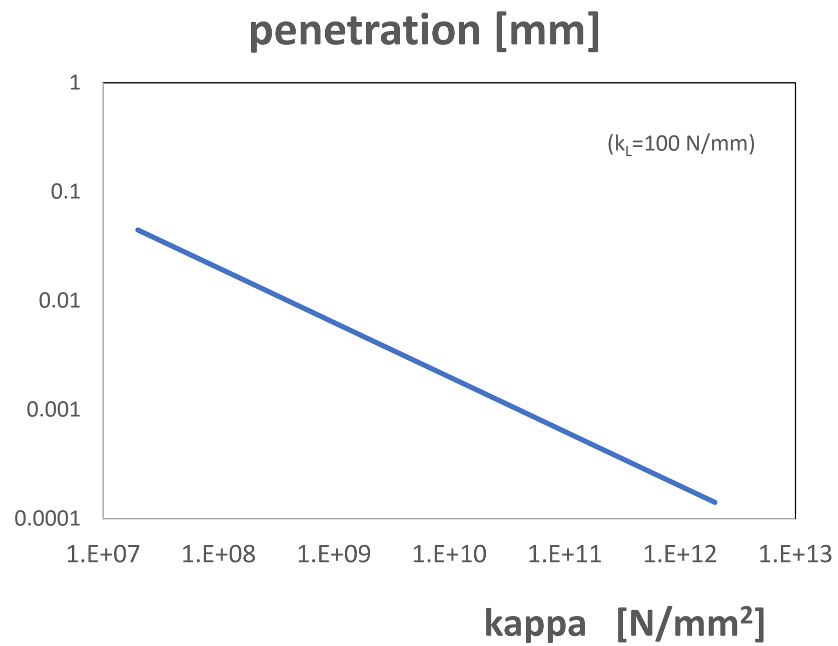
FORCE IN THE SPRING : (CONTACT ELEMENT)

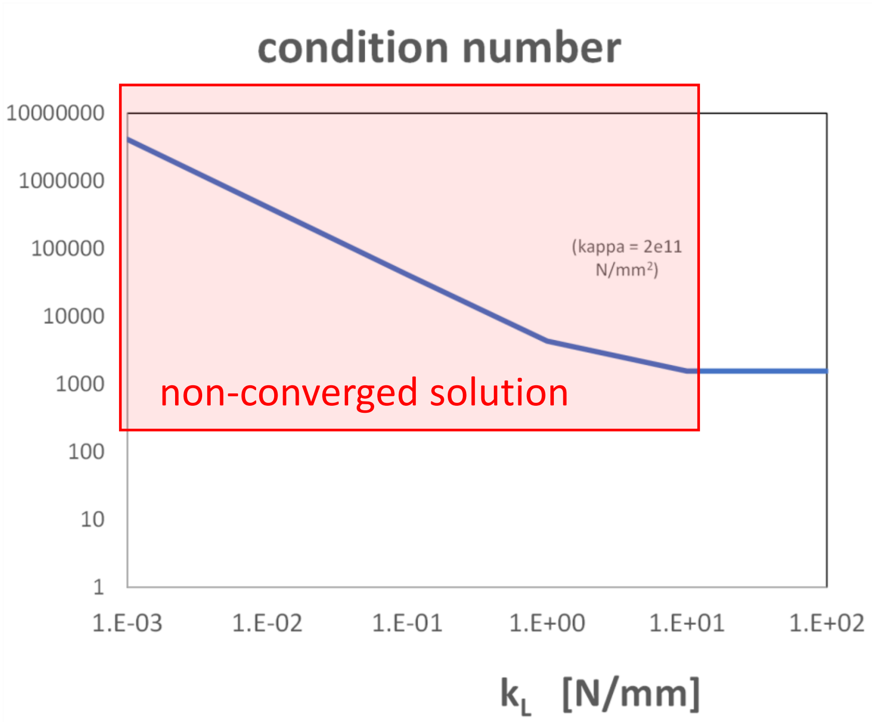
$$F_{c_i} = \Big(\big(k_L + \mathcal{H}(q_2 - q_3)\big) \cdot (q_2 - q_3)\Big)_i =$$

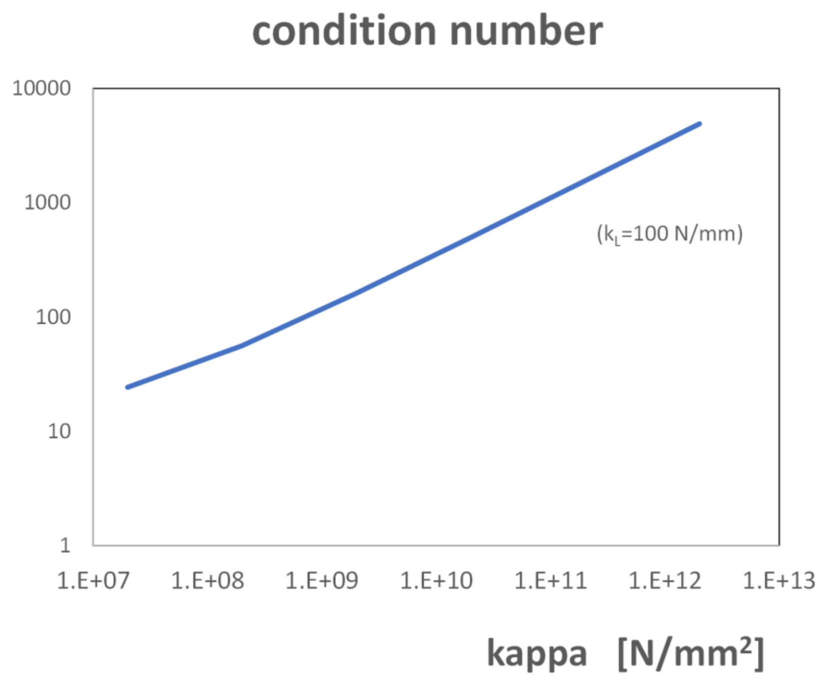
$$= \left(k_L + \mathcal{H} \cdot \text{PENETRATION}\right) \cdot \text{PENETRATION}$$


1.0E+16
1.0E+13
1.0E+10
1.0E+07
1.0E+04
1.0E+01
1.0E-02
1.0E-05
1.0E-08
1.0E-11
1.0E-14
1.0E-17
0
10
20
30
U L2
F L2
U CRIT
F CRIT


penetration [mm]
10
1
0.1
0.01
0.001
0.0001
(kappa = 2e11 N/mm²)
non-converged solution
1.E-03
1.E-02
1.E-01
1.E+00
1.E+01
1.E+02
k_L [N/mm]


penetration [mm]
1
0.1
0.01
0.001
0.0001
(k_L=100 N/mm)
1.E+07
1.E+08
1.E+09
1.E+10
1.E+11
1.E+12
1.E+13
kappa [N/mm²]


condition number
10000000
1000000
100000
10000
1000
100
10
1
(kappa = 2e11 N/mm²)
non-converged solution
1.E-03
1.E-02
1.E-01
1.E+00
1.E+01
1.E+02
k_L [N/mm]


condition number
10000
1000
100
10
1
(k_L=100 N/mm)
1.E+07
1.E+08
1.E+09
1.E+10
1.E+11
1.E+12
1.E+13
kappa [N/mm²]