



Wydział Mechaniczny Energetyki i Lotnictwa
Zakład Wytrzymałości Materiałów i Konstrukcji



Finite element method 2 (FEM2)

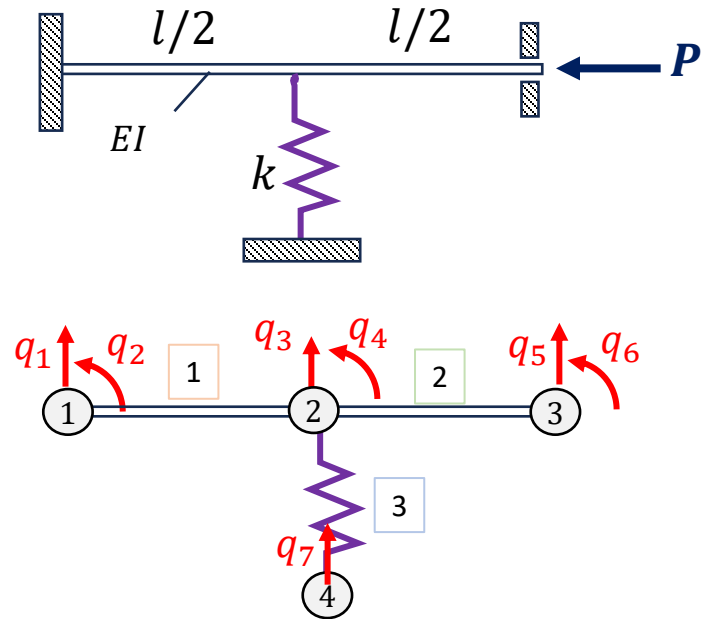
Buckling analysis. Examples

Example 1. Find the critical load for the beam and spring model.

$$([K] - \lambda_* [K_\sigma])\{q\} = 0$$

$$[k]_1 = [k]_2 = \frac{2EI}{l_e^3} \begin{bmatrix} 6 & 3l_e & -6 & 3l_e \\ 3l_e & 2l_e^2 & -3l_e & l_e^2 \\ -6 & -3l_e & 6 & -3l_e \\ 3l_e & l_e^2 & -3l_e & 2l_e^2 \end{bmatrix}$$

$$[k]_3 = \frac{2EI}{l_e^3} \begin{bmatrix} \frac{kl_e^3}{2EI} & -\frac{kl_e^3}{2EI} \\ -\frac{kl_e^3}{2EI} & \frac{kl_e^3}{2EI} \end{bmatrix}$$



Global stiffness matrix:

$$[K] = \frac{2EI}{l_e^3} \begin{bmatrix} 6 & 3l_e & -6 & 3l_e & 0 & 0 & 0 & 0 \\ 0 & 2l_e^2 & -3l_e & l_e^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6+6+\frac{kl_e^3}{2EI} & 3l_e-3l_e & -6 & 3l_e & 0 & -\frac{kl_e^3}{2EI} \\ 0 & 0 & 0 & 2l_e^2+2l_e^2 & -3l_e & l_e^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & -3l_e & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2l_e^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{kl_e^3}{2EI} \end{bmatrix}$$

← Symm. →

Boundary conditions: $q_1 = q_2 = q_5 = q_6 = q_7 = 0$

$$\left(\frac{2EI}{l_e^3} \begin{bmatrix} 12 + \frac{kl_e^3}{2EI} & 0 \\ 0 & 4l_e^2 \end{bmatrix} - \frac{\lambda_*}{30l_e} \begin{bmatrix} 72 & 0 \\ 0 & 8l_e^2 \end{bmatrix} \right) \begin{Bmatrix} q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The auxiliary constants:

$$\lambda = \frac{l_e^2}{60EI} \cdot \lambda_*$$

$$\beta = \frac{kl_e^3}{2EI}$$



$$\begin{bmatrix} 12 + \beta - 72\lambda & 0 \\ 0 & 4l_e^2(1 - 2\lambda) \end{bmatrix} \begin{Bmatrix} q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The determinant is zero if:

$$(12 + \beta - 72\lambda)(1 - 2\lambda) = 0$$

roots:

$$\lambda_1 = \frac{12 + \beta}{72}$$

$$\lambda_2 = \frac{1}{2}$$

If $\beta < 24 \rightarrow k < \frac{48EI}{l_e^3} \rightarrow \lambda_1 < \lambda_2$ (a weak spring)

results in **the first buckling mode**:

$$\lambda_1^* = \frac{60EI}{l_e^2} \lambda_1 = \left(40 + \frac{20}{6} \beta \right) \frac{EI}{l_e^2}$$

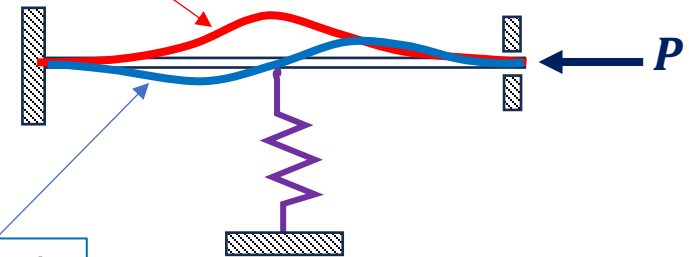
$$[q]_1 = [0, 0, q_3, 0, 0, 0, 0]$$

If $\beta > 24 \rightarrow k > \frac{48EI}{l_e^3} \rightarrow \lambda_2 < \lambda_1$ (a stiff spring)

and **the second buckling mode** occurs:

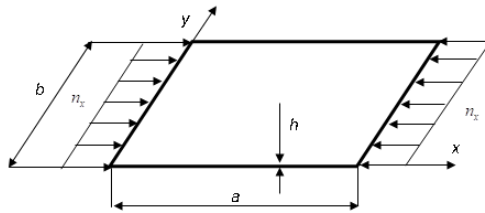
$$\lambda_2^* = \frac{60EI}{l_e^2} \lambda_2 = \frac{30EI}{l_e^2}$$

$$[q]_2 = [0, 0, 0, q_4, 0, 0, 0]$$



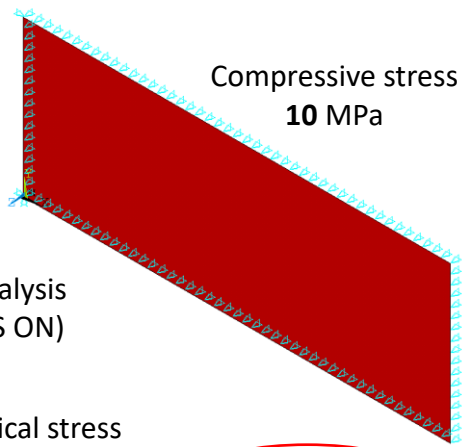
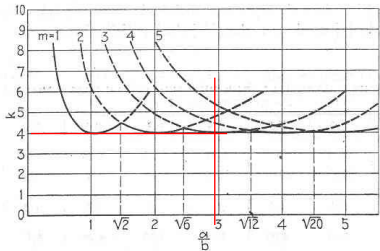
Example 2. Compression of a plate hinged at the edges ($u_z=0$)

$a=885\text{mm}$, $b=302\text{mm}$, $h=2.5\text{mm}$, $E=70000\text{MPa}$, $\nu=0.33$

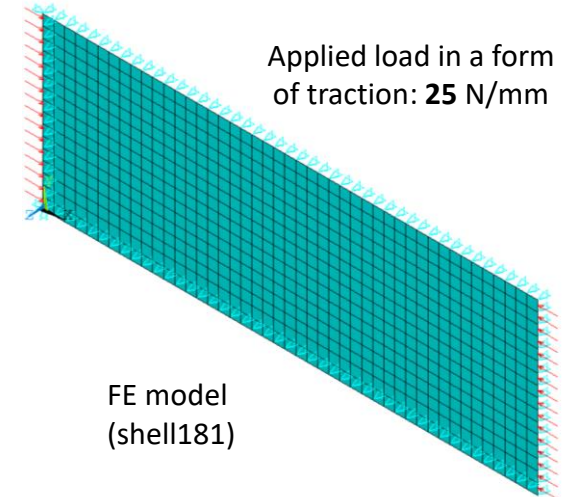


$$\sigma_{cr} = k \frac{\pi^2}{12(1-\nu^2)} \frac{Eh^2}{b^2} \quad k=4 \quad \rightarrow \quad \sigma_{cr} = 17.7 \text{ MPa}$$

(theoretical solution)

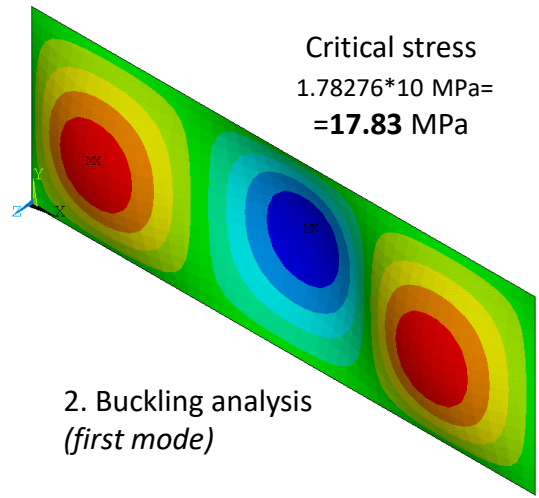


```
ANSYS Rel:
Build 19.1:
NOV 15 20:
13:29:32
PLOT NO.
NODAL SOLU
STEP=1
SUB =1
TIME=1
SX
RSYS=0
PowerGraph
EFACET=1
DMK =.127
SMN =-10
SMX =-10
```



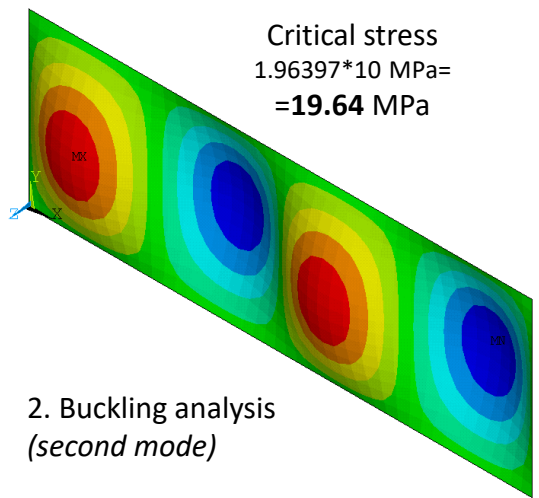
```
Build 19.2
NOV 15 20:2
13:29:17
PLOT NO.
ELEMENTS
PowerGraph
EFACET=1
RSYS=0
PRES-NORM
25
```

1. Static analysis (PRESTRESS ON)



```
DMK =.072755
SMN =-.072695
SMX =.072755
.072695
-.056534
-.040373
-.024211
-.00805
.008111
.024272
.040433
.056594
.072755
```

2. Buckling analysis (first mode)

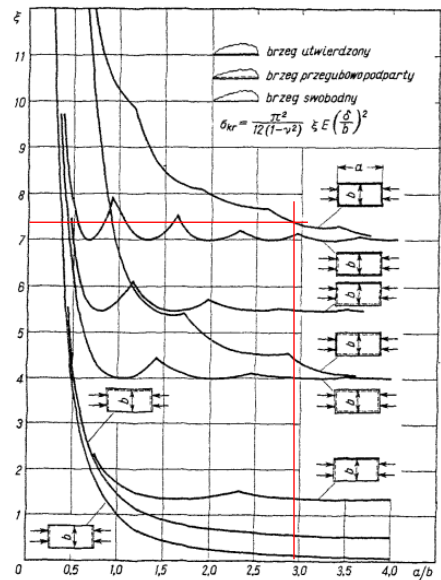


```
15:51:26
PLOT NO. 1
NODAL SOLUTION
STEP=1
SUB =2
FACT=1.96397
UZ (AVG)
RSYS=0
PowerGraphics
EFACET=1
AVRES=Mat
DMK =.054886
SMN =-.054886
SMX =.054886
.054886
-.042689
-.030492
-.018295
-.006098
.006098
.018295
.030492
.042689
.054886
```

2. Buckling analysis (second mode)

Example 3. Compression of a plate hinged at the edges (uz=0)

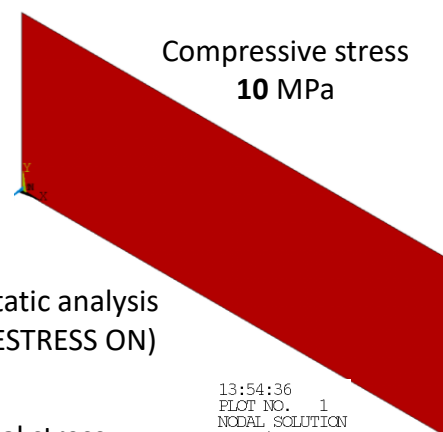
a= 885mm, b= 302mm, h = 2.5mm, E=70000 MPa, v=0.33



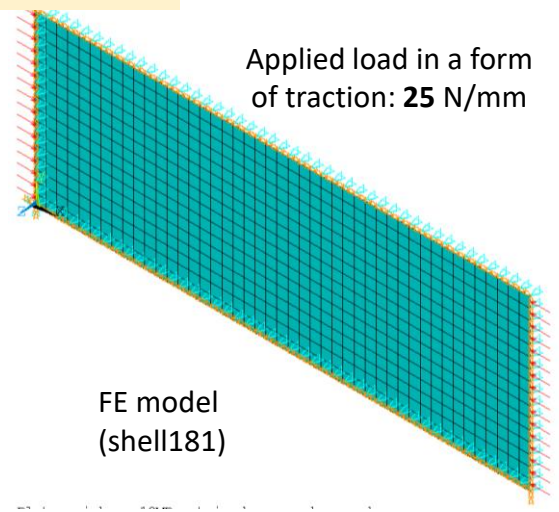
$$\sigma_{cr} = \xi \frac{\pi^2}{12(1-\nu^2)} \frac{Eh^2}{b^2}$$

(theoretical solution)

$\xi = 7.35$ → $\sigma_{cr} = 32.5 \text{ MPa}$



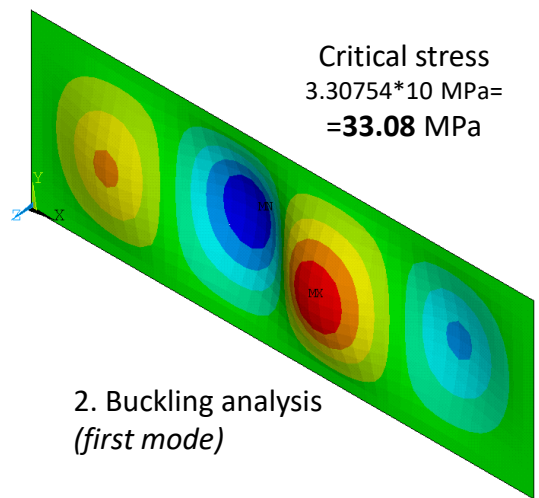
1. Static analysis (PRESTRESS ON)



FE model (shell181)

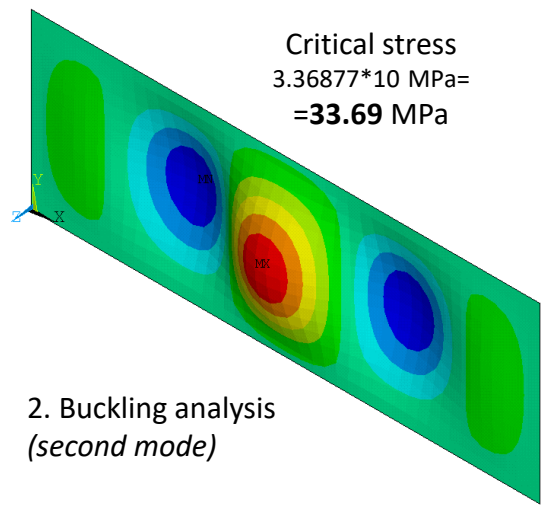
```
Build 19.2
NOV 15 2024
13:53:46
PLOT NO. 1
ELEMENTS
PowerGraphics
EFACET=1
FCT
PRES-NORM
25
```

```
Build 19.2
NOV 15 2024
13:53:58
PLOT NO. 1
NODAL SOLUTION
STEP=2
SUB=1
TIME=2
SX (AVG)
RSYS=0
PowerGraphics
EFACET=1
MAX=-.127208
SMN=-10
SMX=-10
```



2. Buckling analysis (first mode)

```
13:54:36
PLOT NO. 1
NODAL SOLUTION
STEP=1
SUB=1
FACT=3.30754
UZ (AVG)
RSYS=0
PowerGraphics
EFACET=1
AVRES=Mat
DMX=-.070489
SMN=-.070489
SMX=-.070489
-.070489
-.054825
-.03916
-.023496
-.007832
.007832
.023496
.03916
.054825
.070489
```



2. Buckling analysis (second mode)

```
13:54:47
PLOT NO. 1
NODAL SOLUTION
STEP=1
SUB=2
FACT=3.36877
UZ (AVG)
RSYS=0
PowerGraphics
EFACET=1
AVRES=Mat
DMX=-.07841
SMN=-.058848
SMX=-.07841
-.058848
-.043597
-.028346
-.013095
.002156
.017406
.032657
.047908
.063159
.07841
```

Example 4. Shear load in a plate with stringers

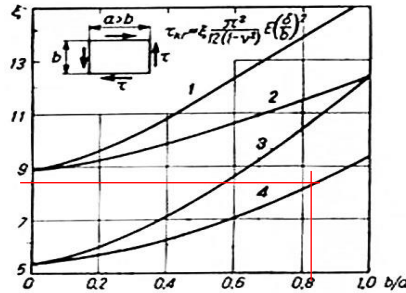
Plate: a=630mm, b=520mm, h=2mm, E=45926 MPa, ν=0.33.

Frame: A_p=1000 mm² (E_p=2·10⁵ MPa, ν_p=0,33)

The structure is loaded by the force F=1000 N.

$$\tau_{cr} = \xi \frac{\pi^2}{12(1-\nu^2)} \frac{Eh^2}{b^2}$$

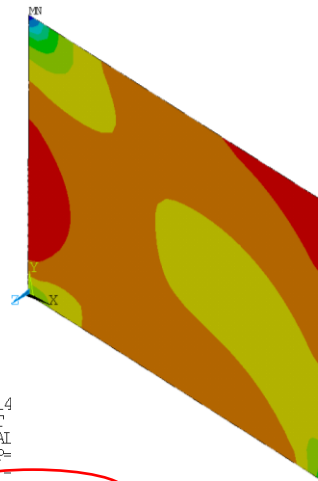
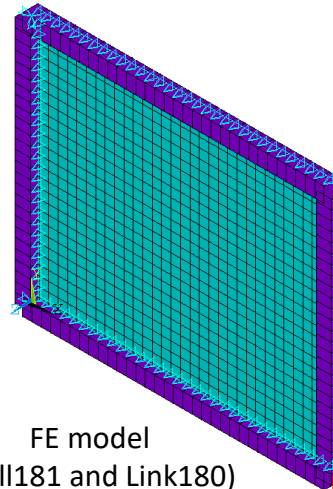
(theoretical solution)



$$\tau_{cr} = 8.25 \frac{\pi^2}{12(1-0.33^2)} \frac{45926 \cdot 2^2}{520^2} = 5.18 \text{ MPa}$$

$$F_{cr} = 5.18 \text{ MPa} \cdot 520 \text{ mm} \cdot 2 \text{ mm} = 5387 \text{ N}$$

1. Static analysis (PRESTRESS ON)



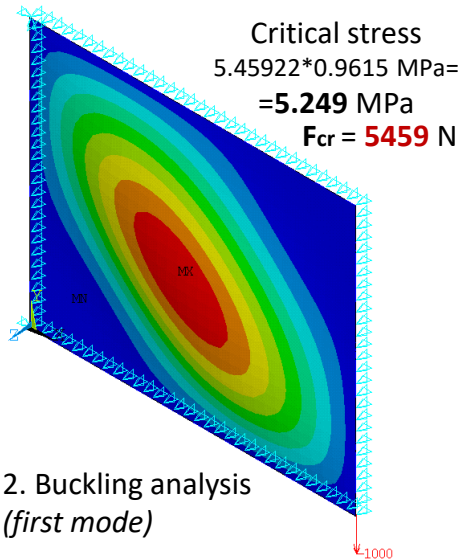
```

*****
NOV 15 2024
15:13:04      1
PLOT NO.    1
NODAL SOLUTION
STEP=1
SUB =1
TIME=1
SKY          (AVG)
RSYS=0
PowerGraphics
EFACET=1
AVRES=Mat
DMX =.039812
SMN =-1.29666
SMX =-.87889

```

Average shear stress:
1000N/520mm/2mm=
=0.9615 MPa

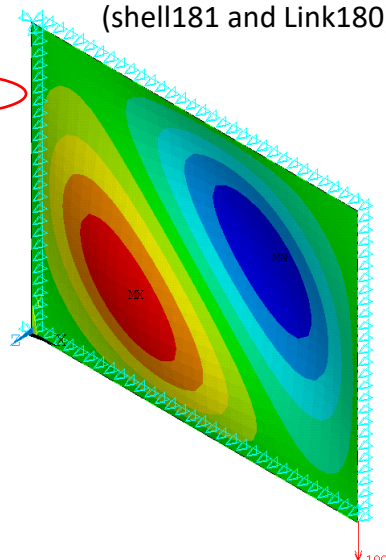
FE model (shell181 and Link180)



```

15:14:37
PLOT NO.    1
NODAL SOLUTION
STEP=1
SUB =1
FACT=5.45922
Uz          (AVG)
RSYS=0
PowerGraphics
EFACET=1
AVRES=Mat
DMX =.49334
SMN =-.014348
SMX =.49334

```



```

15:14
PLOT NO.    1
NODAL SOLUTION
STEP=1
SUB =1
FACT=6.50157
Uz          (AVG)
RSYS=0
PowerGraphics
EFACET=1
AVRES=Mat
DMX =.287132
SMN =-.276329
SMX =.287132

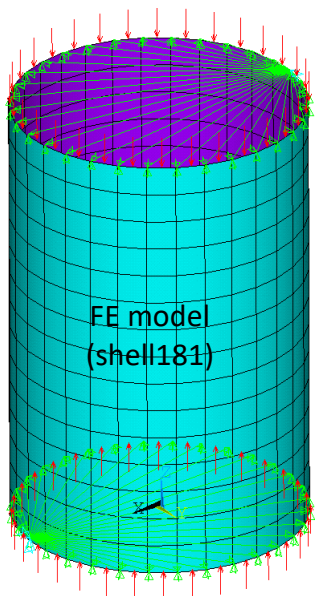
```

2. Buckling analysis (first mode)

2. Buckling analysis (second mode)

Example 5. A cylindrical shell: R=100mm, H=300mm, h=0.5mm, E=7e4 MPa, $\nu=0.33$

Compressive load



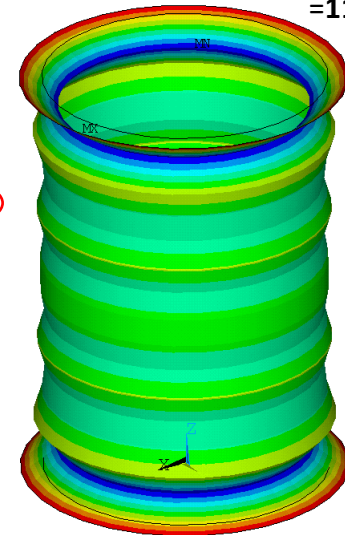
valec sciskany 20MPa



1. Static analysis (PRESTRESS ON)

```
Build 19.2
NOV 15 2024
20:19:10
PLOT NO. 1
NODAL SOLUT:
STEP=1
SUB =1
TIME=1
RSYS=1 (AV)
RSYS=1
PowerGraphic
EFACET=1
AVRES=Mat
DMX =.08623;
SMN =-20
SMX =-20
```

Critical load $5.78937 \times 20 \text{ MPa} = 115.8 \text{ MPa}$

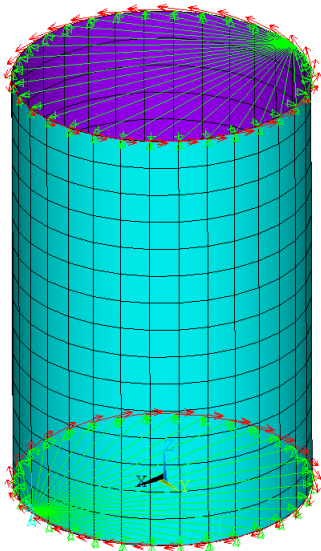


```
NOV 15 2024
20:21:16
PLOT NO. 1
NODAL SOLUTION
STEP=1
SUB =1
FACT=5.78937
UX (AVG)
RSYS=1
PowerGraphics
EFACET=1
AVRES=Mat
DMX =.017603
SMN =-.014135
SMX =.017603
-.014135
-.010608
-.007082
-.003555
-.289E-04
.003497
.007024
.01055
.014077
.017603
```

$\sigma_{kr}^A \approx 210 \text{ MPa}$
 $\sigma_{kr}^B \approx 129.5 \text{ MPa}$
 $\sigma_{kr}^D \approx 91 \text{ MPa}$
 $\sigma_{kr}^{D1} \approx 66.5 \text{ MPa}$

2. Buckling analysis (first mode)

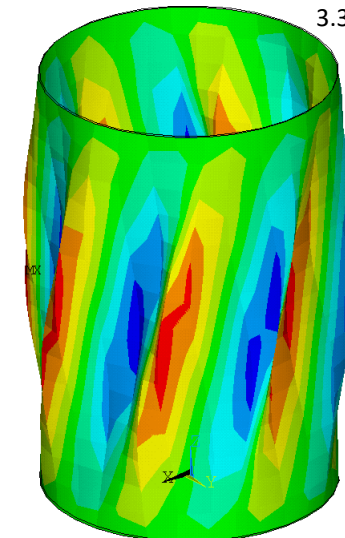
Torsion load



1. Static analysis (PRESTRESS ON)

```
NOV 15 2024
20:25:42
PLOT NO. 1
NODAL SOLUT:
STEP=1
SUB =1
TIME=1
SKY (AV)
MIDDLE
RSYS=SOLO
PowerGraphic
EFACET=1
AVRES=Mat
DMX =.230487
SMN =20.1208
SMX =20.1208
```

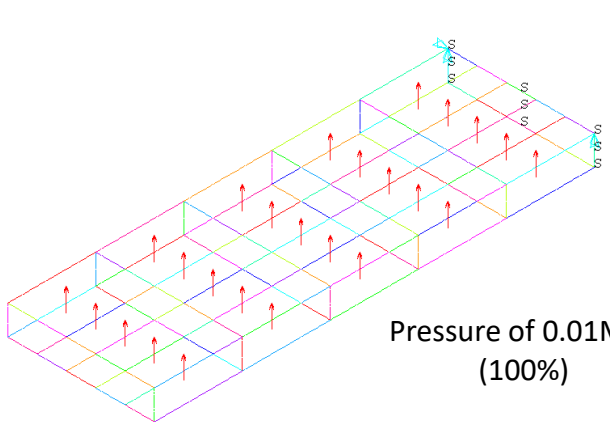
Critical load $3.37898 \times 20 \text{ MPa} = 67.6 \text{ MPa}$



```
NOV 15 2024
20:26:20
PLOT NO. 1
NODAL SOLUTION
STEP=1
SUB =1
FACT=3.37898
UX (AVG)
RSYS=SOLO
PowerGraphics
EFACET=1
AVRES=Mat
DMX =.037928
SMN =-.037921
SMX =.037921
-.037921
-.029494
-.021067
-.01264
-.004213
.004214
.012641
.021068
.029495
.037921
```

2. Buckling analysis (first mode)

Example 6a. Bending of a caisson without stringers : L=1500mm, B=300, H=100, G=0.5, E=7e4 MPa, ν=0.33



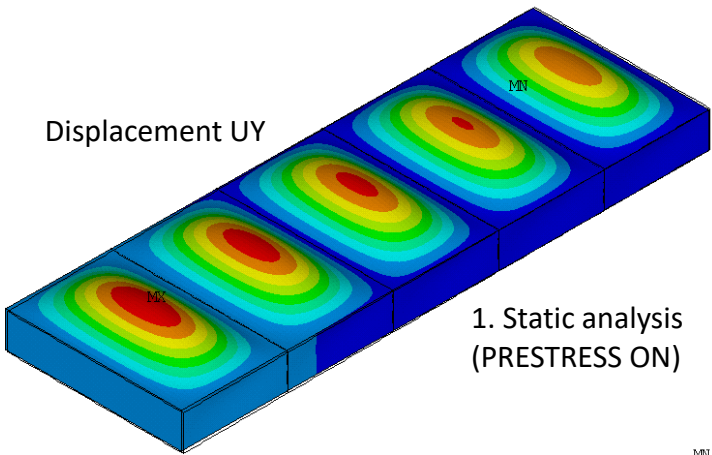
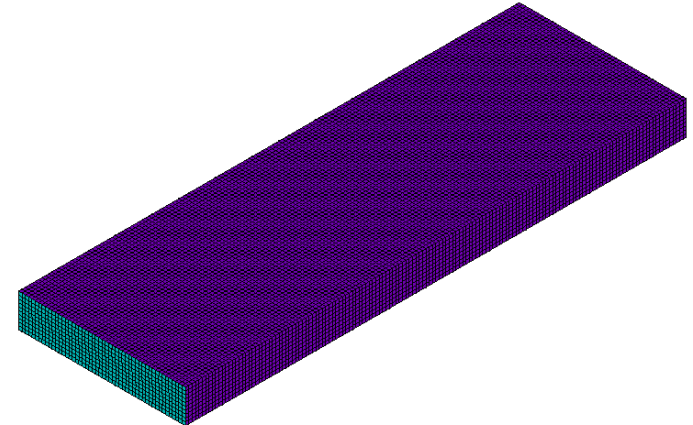
Pressure of 0.01MPa
(100%)

- ```

TYPE NUM
0
PRES-NORM
-.01

*SET,H,100 ! Height
*SET,B,500 ! width
*SET,L,1500 ! length
*SET,n_st,0 ! number of stringers
*SET,th,0.5 ! thickness of cover
*SET,R_st,5 ! radius of stringer
*SET,th_r,2 ! thickness of rib
*SET,E_SIZE,10 ! size of elements
*SET,F_SILA,1000 ! vertical force
*SET,pressure,0.01
*SET,EE,7e5 *SET,NI,,32

```

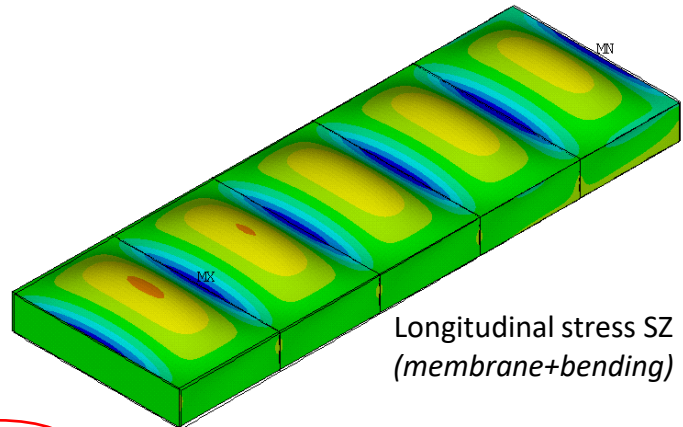


Displacement UY

1. Static analysis  
(PRESTRESS ON)

- ```

TIME=1
UY (AVG)
RSYS=0
PowerGraphics
EFACET=1
AVRES=Mat
DMX =27.7487
SMN =-.499023
SMX =27.7483
- .499023
2.63957
5.77816
8.91675
12.0553
15.1939
18.3325
21.4711
24.6097
27.7483
    
```



Longitudinal stress SZ
(membrane+bending)

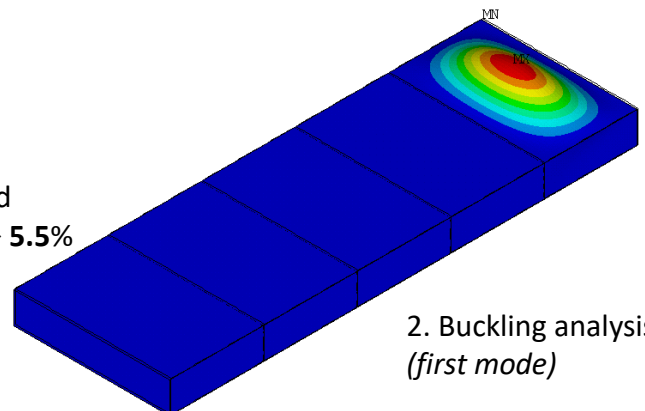
- ```

TIME=1
SZ (AVG)
RSYS=0
PowerGraphics
EFACET=1
AVRES=Mat
DMX =27.7487
SMN =-1734.4
SMX =1559.35
-1734.4
-1368.43
-1002.46
-636.484
-270.511
95.462
461.435
827.409
1193.38
1559.35

```

Keson bez pasow

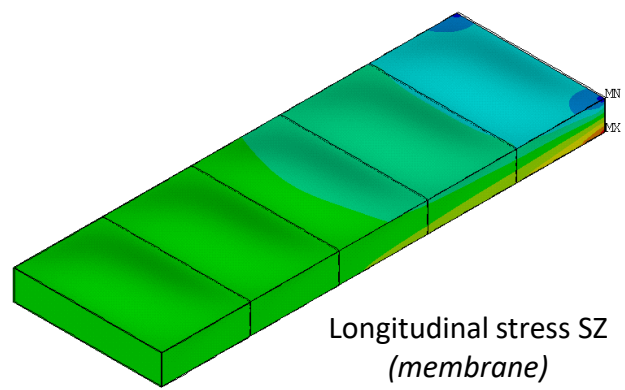
Critical load  
.05534\*.01 MPa → 5.5%



2. Buckling analysis  
(first mode)

- ```

FACT=.055339
CSIM (AVG)
RSYS=0
PowerGraphics
EFACET=1
AVRES=Mat
DMX =.057036
SMX =.057036
0
.006337
.012675
.019012
.025349
.031686
.038024
.044361
.050698
.057036
    
```



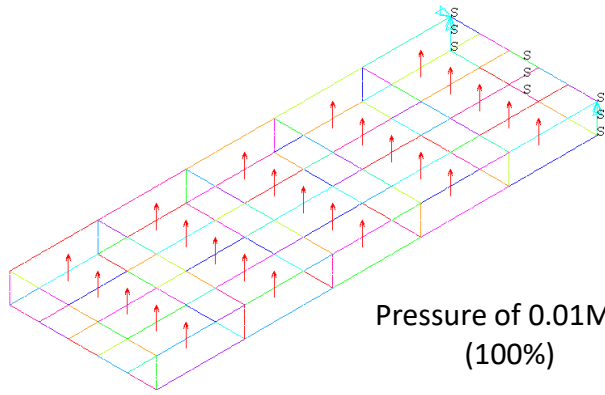
Longitudinal stress SZ
(membrane)

- ```

TIME=1
SZ (AVG)
MIDDLE
RSYS=0
PowerGraphics
EFACET=1
AVRES=Mat
DMX =27.7487
SMN =-383.437
SMX =366.098
-383.437
-300.155
-216.874
-133.592
-50.3104
32.9713
116.253
199.535
282.816
366.098

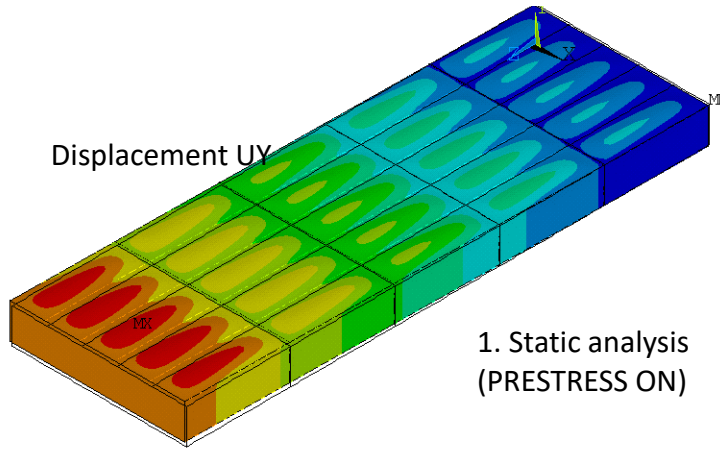
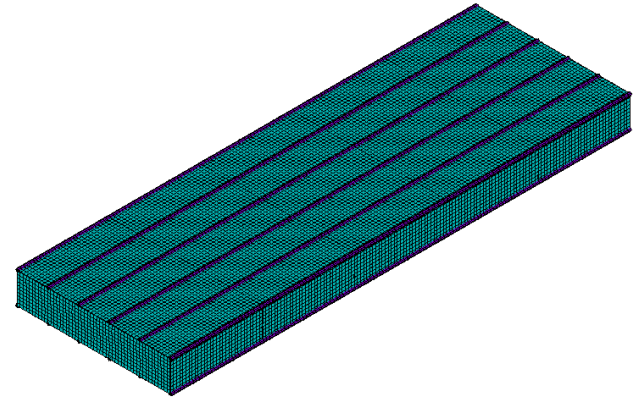
```

# Example 6b. Bending of a caisson with stringers: L=1500mm, B=300, H=100, G=0.5, E=7e4 MPa, $\nu=0.33$



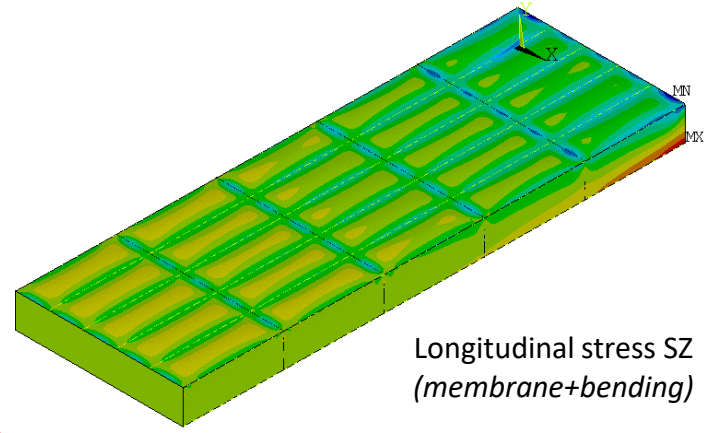
Pressure of 0.01MPa  
(100%)

- TYPE NUM \*SET,H,100 ! Height
- 0 \*SET,B,500 ! width
- PRES-NORM -.01
- \*SET,L,1500 ! length
- \*SET,n\_st,12 ! number of stringers
- \*SET,th,0.5 ! thickness of cover
- \*SET,R\_st,5 ! radius of stringer
- \*SET,th\_r,2 ! thickness of rib
- \*SET,E\_SIZE,10 ! size of elements
- \*SET,F\_SILA,1000 ! vertical force
- \*SET,pressure,0.01
- \*SET,EE,7e5 \*SET,NI,.32

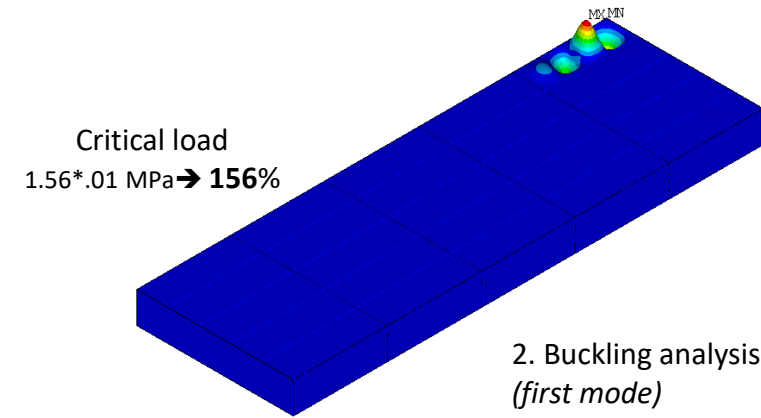


1. Static analysis  
(PRESTRESS ON)

- ```
TIME=1
UY (AVG)
RSYS=0
PowerGraphics
EFACET=1
AVRES=Mat
DMX =1.95444
SMX =1.95374
```
- | |
|---------|
| 0 |
| .217082 |
| .434164 |
| .651247 |
| .868329 |
| 1.08541 |
| 1.30249 |
| 1.51958 |
| 1.73666 |
| 1.95374 |

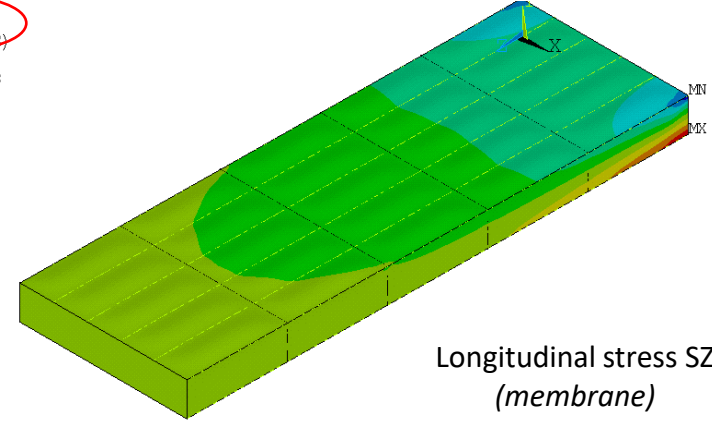


- ```
TIME=1
SZ (AVG)
RSYS=0
PowerGraphics
EFACET=1
AVRES=Mat
DMX =1.95444
SMN =-188.84
SMX =120.946
```
- |          |
|----------|
| -188.84  |
| -154.419 |
| -119.998 |
| -85.5776 |
| -51.1569 |
| -16.7363 |
| 17.6844  |
| 52.1051  |
| 86.5258  |
| 120.946  |



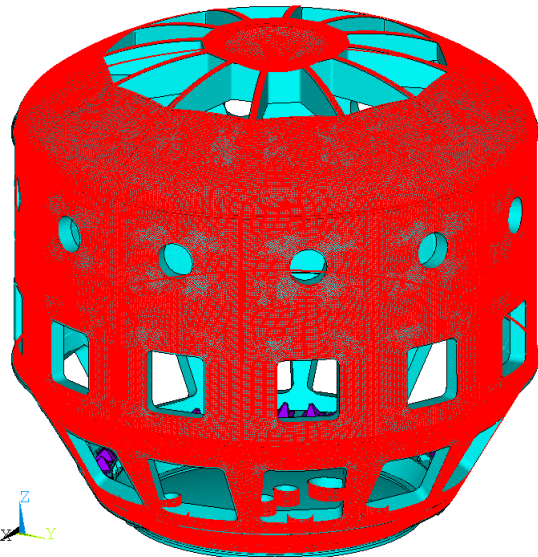
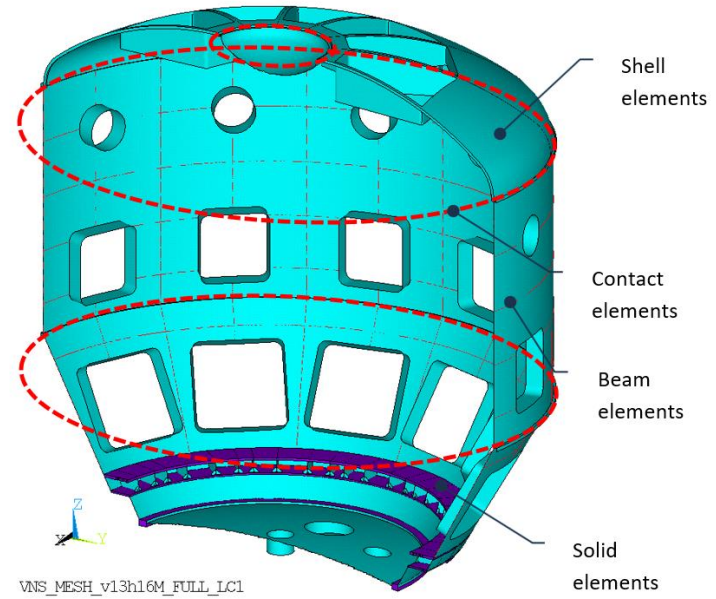
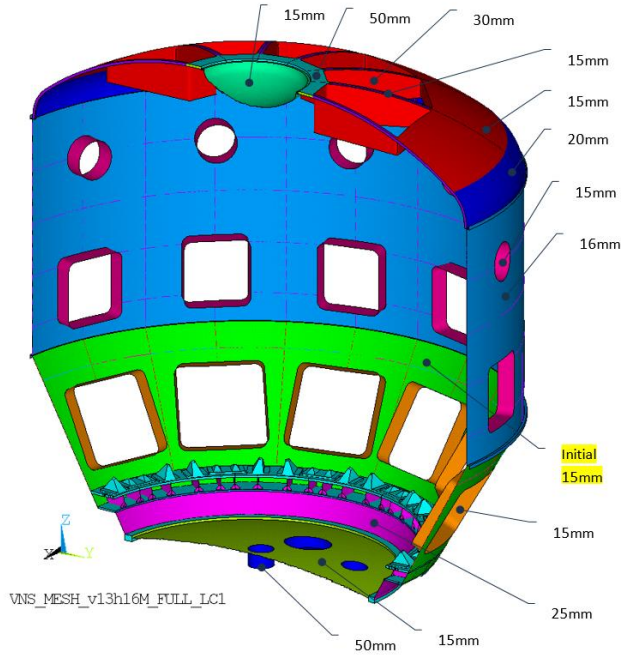
2. Buckling analysis  
(first mode)

- ```
FACT=1.56027
USCM (AVG)
RSYS=0
PowerGraphics
EFACET=1
AVRES=Mat
DMX =.063554
SMX =.063554
```
- | |
|---------|
| 0 |
| .007062 |
| .014123 |
| .021185 |
| .028246 |
| .035308 |
| .042369 |
| .049431 |
| .056492 |
| .063554 |

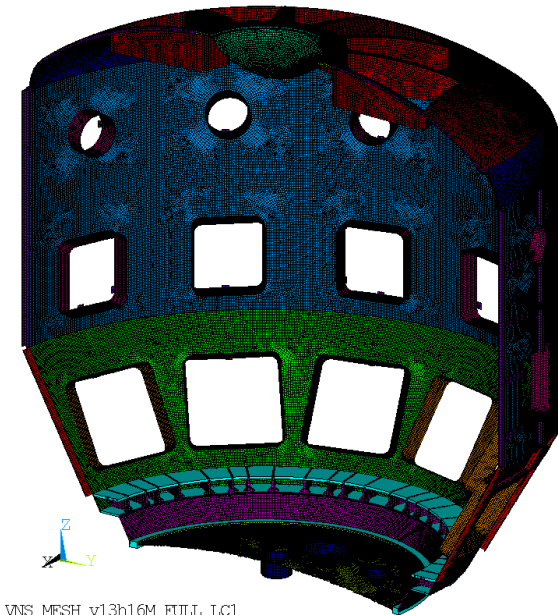


- ```
TIME=1
SZ (AVG)
MIDDLE
RSYS=0
PowerGraphics
EFACET=1
AVRES=Mat
DMX =1.95444
SMN =-160.103
SMX =117.103
```
- |          |
|----------|
| -160.103 |
| -129.302 |
| -98.5015 |
| -67.7008 |
| -36.9002 |
| -6.0995  |
| 24.7012  |
| 55.5018  |
| 86.3025  |
| 117.103  |

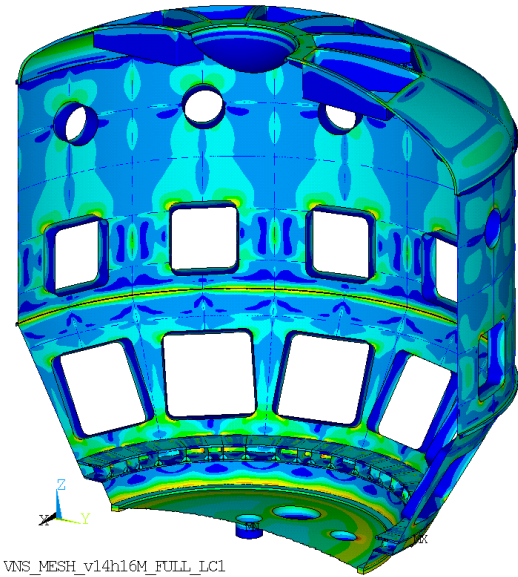
# Example 7. FEM analysis of a cryostat (VNS Feasibility study 2024)



NOV 2 2024  
09:26:50  
PLOT NO. 1  
ELEMENTS  
PowerGraphics  
EFACET=1  
MAT NUM  
PRES-NORM  
.1



# Load case 1: Normal operation (P + D)



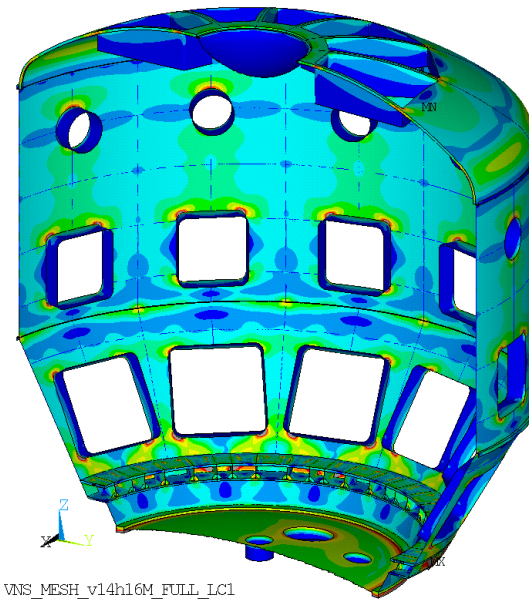
ANSYS 2024 R2  
 Build 24.2  
 NOV 2 2024  
 16:18:12  
 PLOT NO. 1  
 NODAL SOLUTION  
 STEP=1  
 SUB =1  
 TIME=1  
 SECV (AVG)  
 PowerGraphics  
 EFACET=1  
 AVRES=Mat  
 DMX =19.7279  
 SMN =-.087978  
 SMX =280.753

|         |
|---------|
| 0       |
| 19.1111 |
| 38.2222 |
| 57.3333 |
| 76.4444 |
| 95.5556 |
| 114.667 |
| 133.778 |
| 152.889 |
| 172     |

VNS\_MESH\_v14h16M\_FULL\_LC1

Von Mises stress (Pm+Pb) [MPa]

Load case 1: Normal operation (P + D)



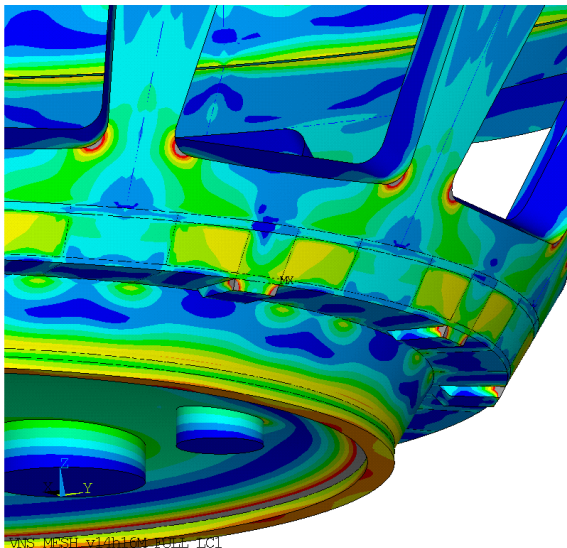
ANSYS 2024 R2  
 Build 24.2  
 NOV 2 2024  
 16:18:16  
 PLOT NO. 1  
 NODAL SOLUTION  
 STEP=1  
 SUB =1  
 TIME=1  
 SECV (AVG)  
 MIDDLE  
 PowerGraphics  
 EFACET=1  
 AVRES=Mat  
 DMX =19.7279  
 SMN =.052089  
 SMX =280.753

|         |
|---------|
| 0       |
| 12.7778 |
| 25.5556 |
| 38.3333 |
| 51.1111 |
| 63.8889 |
| 76.6667 |
| 89.4444 |
| 102.222 |
| 115     |

VNS\_MESH\_v14h16M\_FULL\_LC1

Von Mises membrane stress (Pm) [MPa]

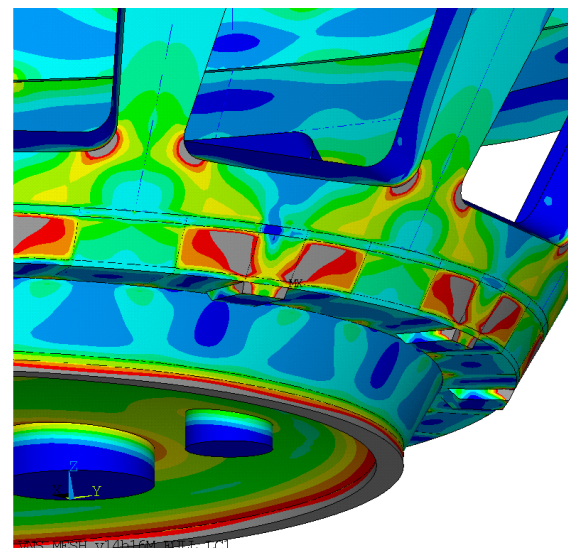
Load case 1: Normal operation (P + D)



Build 24.2  
 NOV 2 2024  
 16:29:25  
 PLOT NO. 1  
 NODAL SOLUTION  
 STEP=1  
 SUB =1  
 TIME=1  
 SECV (AVG)  
 PowerGraphics  
 EFACET=1  
 AVRES=Mat  
 DMX =19.7279  
 SMN =-.087978  
 SMX =281.608

|         |
|---------|
| 0       |
| 19.1111 |
| 38.2222 |
| 57.3333 |
| 76.4444 |
| 95.5556 |
| 114.667 |
| 133.778 |
| 152.889 |
| 172     |

VNS\_MESH\_v14h16M\_FULL\_LC1

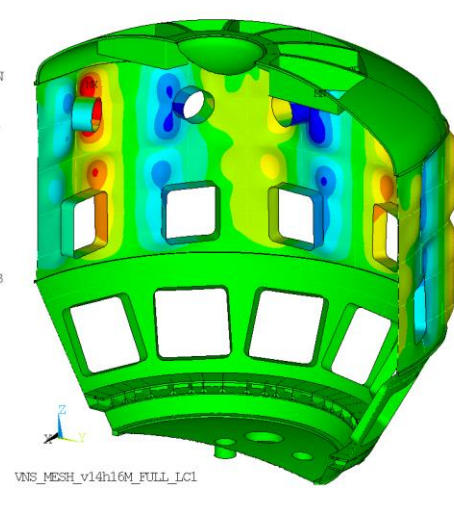
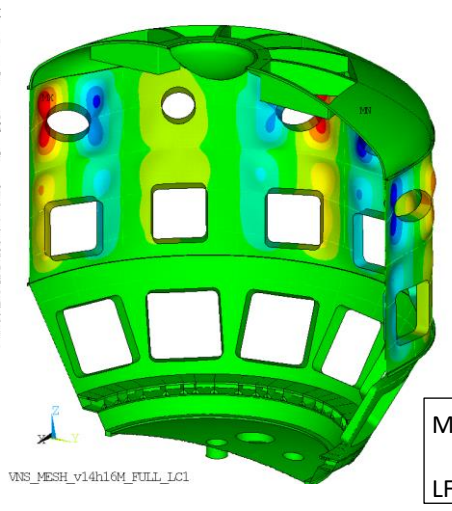
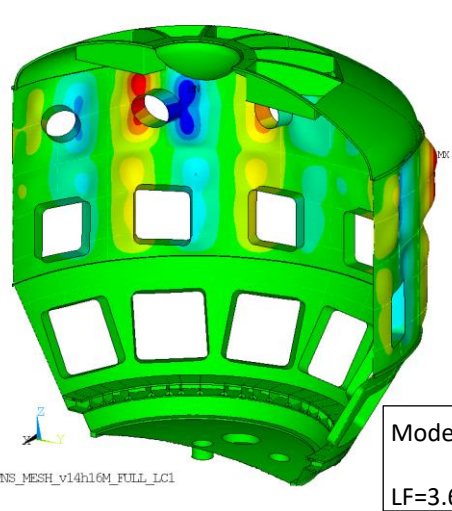
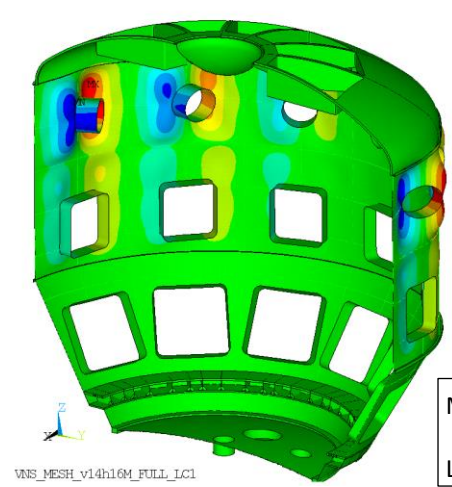
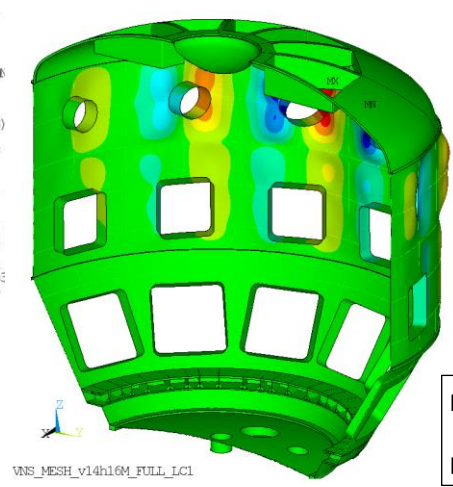
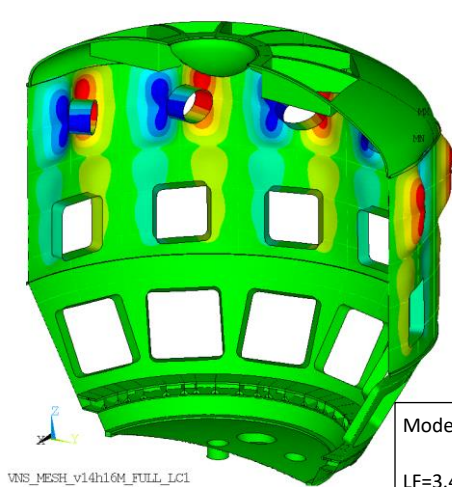


Build 24.2  
 NOV 2 2024  
 16:31:05  
 PLOT NO. 1  
 NODAL SOLUTION  
 STEP=1  
 SUB =1  
 TIME=1  
 SECV (AVG)  
 MIDDLE  
 PowerGraphics  
 EFACET=1  
 AVRES=Mat  
 DMX =19.7279  
 SMN =.052075  
 SMX =281.608

|         |
|---------|
| 0       |
| 12.7778 |
| 25.5556 |
| 38.3333 |
| 51.1111 |
| 63.8889 |
| 76.6667 |
| 89.4444 |
| 102.222 |
| 115     |

VNS\_MESH\_v14h16M\_FULL\_LC1

# Buckling modes 1-6 for Load case 1: Normal operation (P + D)



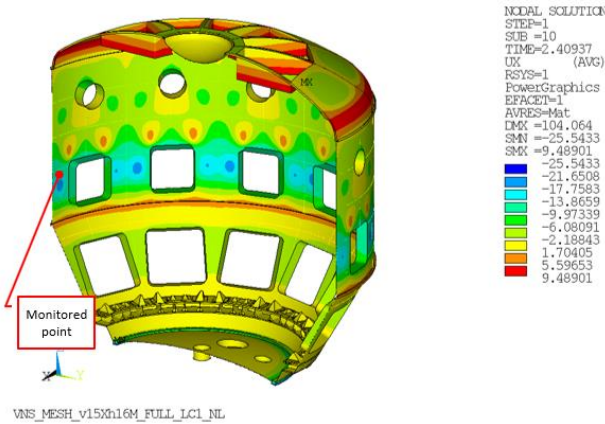
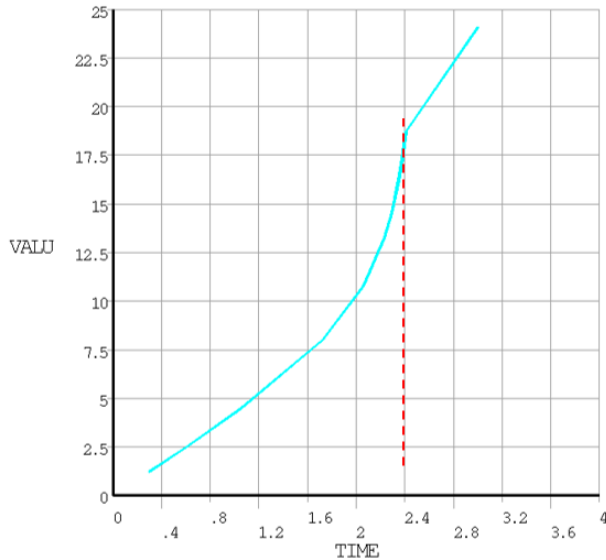
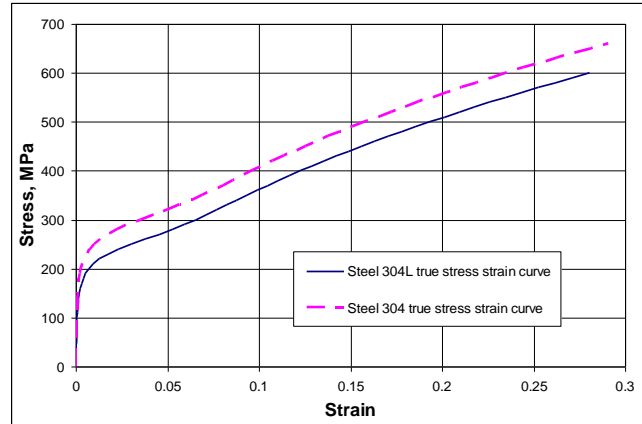
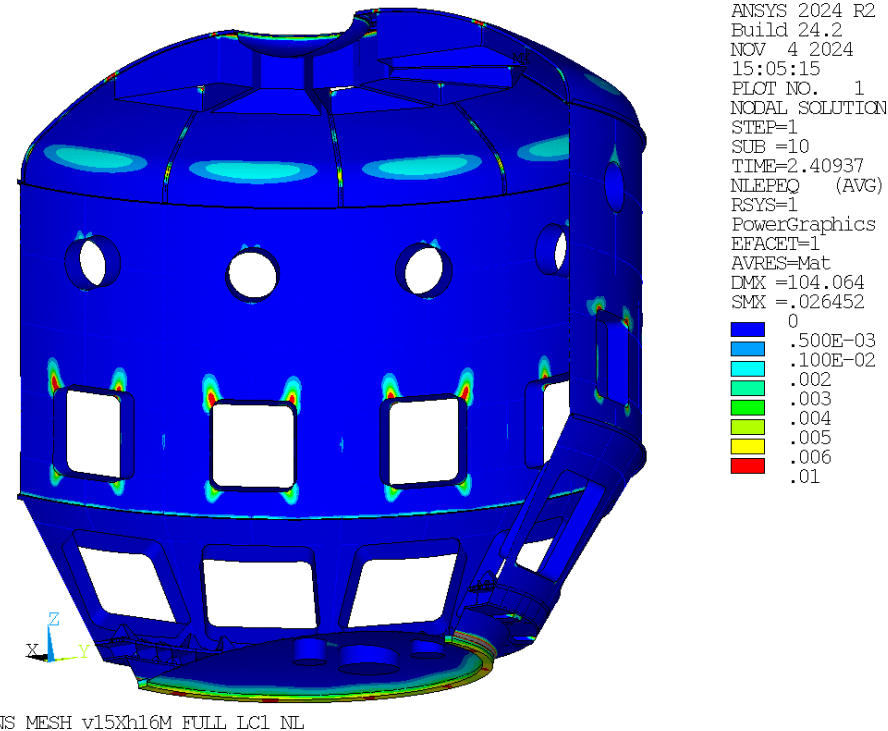


Figure 5-9 Radial displacements [mm] for LF=2.41- Final model: Elastic – Plastic Analysis 2.4 (P + D)



VNS\_MESH\_v15Xh16M\_FULL\_LC1\_NL

Figure 5-11 Radial displacement of the monitored point as a functions of the Load Factor - Final model - Elastic – Plastic Analysis 2.4 (P + D)



Accumulated Equivalent plastic strain for LF=2.41- Final model: Elastic – Plastic Analysis 2.4 (P + D)