Inversely, the material vector dX is the pull back equivalent of the spatial vector dx, which is expressed as*

$$dX = \phi_*^{-1}[dx] = F^{-1}dx. \tag{4.13}$$

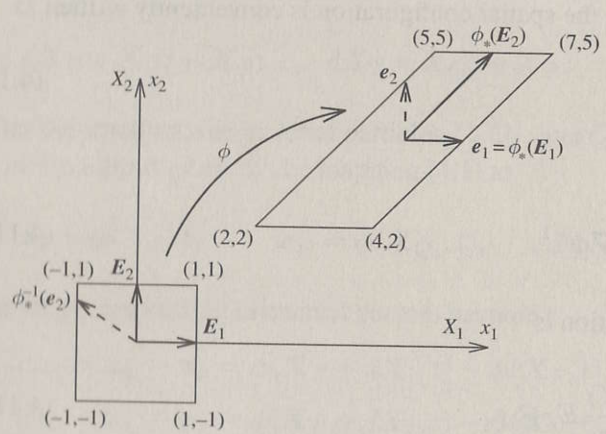
Observe that in (4.12) the nomenclature $\phi_*[\,]$ implies an *operation* that will be evaluated in different ways for different operands $[\,]$.

EXAMPLE 4.2: Uniform deformation

This example illustrates the role of the deformation gradient tensor $\boldsymbol{F}$. Consider the uniform deformation given by the mapping

$$x_1 = \frac{1}{4}(18 + 4X_1 + 6X_2);$$
$$x_2 = \frac{1}{4}(14 + 6X_2);$$

which, for a square of side 2 units initially centred at $\boldsymbol{X} = (0,0)$, produces the deformation shown below.



$$\boldsymbol{F} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 3 \\ 0 & 3 \end{bmatrix}; \quad \boldsymbol{F}^{-1} = \frac{1}{3}\begin{bmatrix} 3 & -3 \\ 0 & 2 \end{bmatrix}.$$

(continued)

* In the literature $\phi_*[\,]$ and $\phi_*^{-1}[\,]$ are often written as ϕ_* and ϕ^* respectively.

EXAMPLE 4.2: *(cont.)*

Unit vectors $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ in the initial configuration deform to

$$\phi_*[\boldsymbol{E}_1] = \boldsymbol{F}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \phi_*[\boldsymbol{E}_2] = \boldsymbol{F}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix};$$

and unit vectors in the current (deformed) configuration deform from

$$\phi_*^{-1}[\mathrm{e}_1] = \boldsymbol{F}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \phi_*^{-1}[\mathrm{e}_2] = \boldsymbol{F}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{2}{3} \end{bmatrix}.$$

4.5 STRAIN

As a general measure of deformation, consider the change in the scalar product of the two elemental vectors $d\boldsymbol{X}_1$ and $d\boldsymbol{X}_2$ shown in Figure 4.2 as they deform to $d\boldsymbol{x}_1$ and $d\boldsymbol{x}_2$. This change will involve both the stretching (that is, change in length) and changes in the enclosed angle between the two vectors. Recalling Equation (4.7a,b), the spatial scalar product $d\boldsymbol{x}_1 \cdot d\boldsymbol{x}_2$ can be found in terms of the material vectors $d\boldsymbol{X}_1$ and $d\boldsymbol{X}_2$ as

$$d\boldsymbol{x}_1 \cdot d\boldsymbol{x}_2 = d\boldsymbol{X}_1 \cdot \boldsymbol{C}\, d\boldsymbol{X}_2, \tag{4.14}$$

where $\boldsymbol{C}$ is the *right Cauchy–Green deformation tensor*, which is given in terms of the deformation gradient as $\boldsymbol{F}$ as

$$\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}. \tag{4.15}$$

Note that in Equation (4.15) the tensor $\boldsymbol{C}$ operates on the material vectors $d\boldsymbol{X}_1$ and $d\boldsymbol{X}_2$ and consequently $\boldsymbol{C}$ is called a material tensor quantity.

Alternatively, the initial material scalar product $d\boldsymbol{X}_1 \cdot d\boldsymbol{X}_2$ can be obtained in terms of the spatial vectors $d\boldsymbol{x}_1$ and $d\boldsymbol{x}_2$ via the *left Cauchy–Green* or *Finger tensor* $\boldsymbol{b}$ as*

$$d\boldsymbol{X}_1 \cdot d\boldsymbol{X}_2 = d\boldsymbol{x}_1 \cdot \boldsymbol{b}^{-1} d\boldsymbol{x}_2, \tag{4.16}$$

where $\boldsymbol{b}$ is

$$\boldsymbol{b} = \boldsymbol{F}\boldsymbol{F}^T. \tag{4.17}$$

* In $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$, $\boldsymbol{F}$ is on the right and in $\boldsymbol{b} = \boldsymbol{F}\boldsymbol{F}^T$, $\boldsymbol{F}$ is on the left.

Observe that in Equation (4.16) $\boldsymbol{b}^{-1}$ operates on the spatial vectors $d\boldsymbol{x}_1$ and $d\boldsymbol{x}_2$, and consequently $\boldsymbol{b}^{-1}$, or indeed $\boldsymbol{b}$ itself, is a spatial tensor quantity.

The change in scalar product can now be found in terms of the material vectors $d\boldsymbol{X}_1$ and $d\boldsymbol{X}_2$ and the *Lagrangian* or *Green strain tensor* $\boldsymbol{E}$ as

$$\frac{1}{2}(d\boldsymbol{x}_1 \cdot d\boldsymbol{x}_2 - d\boldsymbol{X}_1 \cdot d\boldsymbol{X}_2) = d\boldsymbol{X}_1 \cdot \boldsymbol{E}\, d\boldsymbol{X}_2, \tag{4.18a}$$

where the material tensor $\boldsymbol{E}$ is

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I}). \tag{4.18b}$$

Alternatively, the same change in scalar product can be expressed with reference to the spatial elemental vectors $d\boldsymbol{x}_1$ and $d\boldsymbol{x}_2$ and the *Eulerian* or *Almansi strain tensor* $\boldsymbol{e}$ as

$$\frac{1}{2}(d\boldsymbol{x}_1 \cdot d\boldsymbol{x}_2 - d\boldsymbol{X}_1 \cdot d\boldsymbol{X}_2) = d\boldsymbol{x}_1 \cdot \boldsymbol{e}\, d\boldsymbol{x}_2, \tag{4.19a}$$

where the spatial tensor $\boldsymbol{e}$ is

$$\boldsymbol{e} = \frac{1}{2}(\boldsymbol{I} - \boldsymbol{b}^{-1}). \tag{4.19b}$$

EXAMPLE 4.3: Green and Almansi strain tensors

For the deformation given in Example 4.2 the right and left Cauchy–Green deformation tensors are respectively,

$$\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F} = \frac{1}{2}\begin{bmatrix} 2 & 3 \\ 3 & 9 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \boldsymbol{F}\boldsymbol{F}^T = \frac{1}{4}\begin{bmatrix} 13 & 9 \\ 9 & 9 \end{bmatrix},$$

from which the Green's strain tensor is simply

$$\boldsymbol{E} = \frac{1}{4}\begin{bmatrix} 0 & 3 \\ 3 & 7 \end{bmatrix};$$

and the Almansi strain tensor is,

$$\boldsymbol{e} = \frac{1}{18}\begin{bmatrix} 0 & 9 \\ 9 & -4 \end{bmatrix}.$$

The physical interpretation of these strain measures will be demonstrated in the next example.

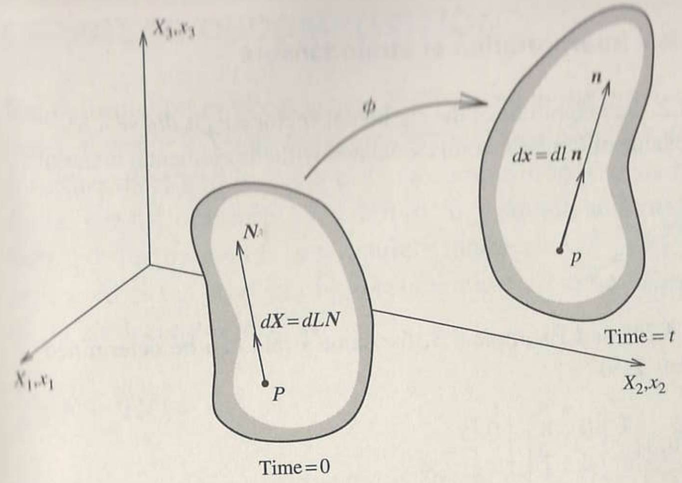


FIGURE 4.3 Change in length.

Remark 4.3: The general nature of the scalar product as a measure of deformation can be clarified by taking $d\boldsymbol{X}_2$ and $d\boldsymbol{X}_1$ equal to $d\boldsymbol{X}$ and consequently $d\boldsymbol{x}_1 = d\boldsymbol{x}_2 = d\boldsymbol{x}$. This enables initial (material) and current (spatial) elemental lengths squared to be determined as (Figure 4.3)

$$dL^2 = d\boldsymbol{X} \cdot d\boldsymbol{X}; \qquad dl^2 = d\boldsymbol{x} \cdot d\boldsymbol{x}. \tag{4.20a,b}$$

The change in the squared lengths that occurs as the body deforms from the initial to the current configuration can now be written in terms of the elemental material vector $d\boldsymbol{X}$ as

$$\frac{1}{2}(dl^2 - dL^2) = d\boldsymbol{X} \cdot \boldsymbol{E}\, d\boldsymbol{X}, \tag{4.21}$$

which, upon division by dL^2, gives the scalar Green's strain as

$$\frac{dl^2 - dL^2}{2\,dL^2} = \frac{d\boldsymbol{X}}{dL} \cdot \boldsymbol{E}\frac{d\boldsymbol{X}}{dL}, \tag{4.22}$$

where $d\boldsymbol{X}/dL$ is a unit material vector $\boldsymbol{N}$ in the direction of $d\boldsymbol{X}$, hence, finally

$$\frac{1}{2}\left(\frac{dl^2 - dL^2}{dL^2}\right) = \boldsymbol{N} \cdot \boldsymbol{E}\boldsymbol{N}. \tag{4.23}$$

Using Equation (4.19a), a similar expression involving the Almansi strain tensor can be derived as

$$\frac{1}{2}\left(\frac{dl^2 - dL^2}{dl^2}\right) = \boldsymbol{n} \cdot \boldsymbol{e}\boldsymbol{n}, \tag{4.24}$$

where $\boldsymbol{n}$ is a unit vector in the direction of $d\boldsymbol{x}$.

EXAMPLE 4.4: Physical interpretation of strain tensors

Referring to Example 4.2, the magnitude of the elemental vector $d\boldsymbol{x}_2$ is $dl_2 = 4.5^{1/2}$. Using (4.23), the scalar value of Green's strain associated with the elemental material vector $d\boldsymbol{X}_2$ is

$$\varepsilon_G = \frac{1}{2}\left(\frac{dl^2 - dL^2}{dL^2}\right) = \frac{7}{4}.$$

Again using Equation (4.23) and Example 4.3, the same strain can be determined from Green's strain tensor $\boldsymbol{E}$ as

$$\varepsilon_G = \boldsymbol{N}^T\boldsymbol{E}\boldsymbol{N} = [0,1]\frac{1}{4}\begin{bmatrix}0 & 3\\ 3 & 7\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \frac{7}{4}.$$

Using Equation (4.24) the scalar value of the Almansi strain associated with the elemental spatial vector $d\boldsymbol{x}_2$ is

$$\varepsilon_A = \frac{1}{2}\left(\frac{dl^2 - dL^2}{dl^2}\right) = \frac{7}{18}.$$

Alternatively, again using Equation (4.24) and Example 4.3 the same strain is determined from the Almansi strain tensor $\boldsymbol{e}$ as

$$\varepsilon_A = \boldsymbol{n}^T\boldsymbol{e}\boldsymbol{n} = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]\frac{1}{18}\begin{bmatrix}0 & 9\\ 9 & -4\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} = \frac{7}{18}.$$

Remark 4.4: In terms of the language of pull back and push forward, the material and spatial strain measures can be related through the operator ϕ_*. Precisely, how this operator works in this case can be discovered by recognizing, because of their definitions, the equality

$$d\boldsymbol{x}_1 \cdot \boldsymbol{e}\, d\boldsymbol{x}_2 = d\boldsymbol{X}_1 \cdot \boldsymbol{E}\, d\boldsymbol{X}_2 \tag{4.25}$$

for any corresponding pairs of elemental vectors. Recalling Equations (4.12–4.13) enables the push-forward and pull-back operations to be written as

Push forward

$$\boldsymbol{e} = \phi_*[\boldsymbol{E}] = \boldsymbol{F}^{-T}\boldsymbol{E}\boldsymbol{F}^{-1}. \tag{4.26a}$$

Pull back

$$\boldsymbol{E} = \phi_*^{-1}[\boldsymbol{e}] = \boldsymbol{F}^T\boldsymbol{e}\boldsymbol{F}. \tag{4.26b}$$

4.6 POLAR DECOMPOSITION

The deformation gradient tensor $\boldsymbol{F}$ discussed in the previous sections transforms a material vector $d\boldsymbol{X}$ into the corresponding spatial vector $d\boldsymbol{x}$. The crucial role of $\boldsymbol{F}$ is further disclosed in terms of its decomposition into stretch and rotation components. The use of the physical terminology stretch and rotation will become clearer later. For the moment, from a purely mathematical point of view, the tensor $\boldsymbol{F}$ is expressed as the product of a *rotation tensor* $\boldsymbol{R}$ times a *stretch tensor* $\boldsymbol{U}$ to define the polar decomposition as

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U}. \tag{4.27}$$

For the purpose of evaluating these tensors, recall the definition of the right Cauchy–Green tensor $\boldsymbol{C}$ as

$$\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F} = \boldsymbol{U}^T\boldsymbol{R}^T\boldsymbol{R}\boldsymbol{U}. \tag{4.28}$$

Given that $\boldsymbol{R}$ is an orthogonal rotation tensor as defined in Equation (2.26), that is, $\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}$, and choosing $\boldsymbol{U}$ to be a symmetric tensor gives a unique definition of the *material stretch tensor* $\boldsymbol{U}$ in terms of $\boldsymbol{C}$ as

$$\boldsymbol{U}^2 = \boldsymbol{U}\boldsymbol{U} = \boldsymbol{C}. \tag{4.29}$$

In order to actually obtain $\boldsymbol{U}$ from this equation, it is first necessary to evaluate the principal directions of $\boldsymbol{C}$, denoted here by the eigenvector triad $\{\boldsymbol{N}_1, \boldsymbol{N}_2, \boldsymbol{N}_3\}$ and their corresponding eigenvalues λ_1^2, λ_2^2, and λ_3^2, which enable $\boldsymbol{C}$ to be expressed as

$$\boldsymbol{C} = \sum_{\alpha=1}^{3} \lambda_\alpha^2\, \boldsymbol{N}_\alpha \otimes \boldsymbol{N}_\alpha, \tag{4.30}$$

where, because of the symmetry of $\boldsymbol{C}$, the triad $\{\boldsymbol{N}_1, \boldsymbol{N}_2, \boldsymbol{N}_3\}$ are orthogonal unit vectors. Combining Equations (4.29) and (4.30), the material stretch tensor $\boldsymbol{U}$ can be easily obtained as

$$\boldsymbol{U} = \sum_{\alpha=1}^{3} \lambda_\alpha\, \boldsymbol{N}_\alpha \otimes \boldsymbol{N}_\alpha. \tag{4.31}$$

Once the stretch tensor $\boldsymbol{U}$ is known, the rotation tensor $\boldsymbol{R}$ can be easily evaluated from Equation (4.27) as $\boldsymbol{R} = \boldsymbol{F}\boldsymbol{U}^{-1}$.

$$\hat{\boldsymbol{n}}_1 = \hat{\boldsymbol{F}}\boldsymbol{N}_1 = \begin{bmatrix} -0.3366 \\ 1.5856 \\ 0 \end{bmatrix}; \qquad \hat{\boldsymbol{n}}_2 = \hat{\boldsymbol{F}}\boldsymbol{N}_2 = \begin{bmatrix} -0.8453 \\ -0.1794 \\ 0 \end{bmatrix};$$

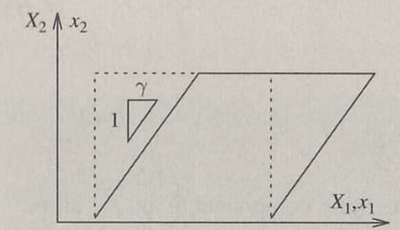
$$\hat{\boldsymbol{n}}_3 = \hat{\boldsymbol{F}}\boldsymbol{N}_3 = \begin{bmatrix} 0 \\ 0 \\ 0.7138 \end{bmatrix}.$$

Since $\boldsymbol{N}_\alpha$ are principal directions, $\hat{\boldsymbol{n}}_\alpha$ are orthogonal vectors and the corresponding elemental spatial volume is conveniently,

$$dv = \|\hat{\boldsymbol{n}}_1\|\|\hat{\boldsymbol{n}}_2\|\|\hat{\boldsymbol{n}}_3\| = 1,$$

thus demonstrating the isochoric nature of $\hat{\boldsymbol{F}}$.

EXAMPLE 4.8: Simple shear



Sometimes the motion of a body is isochoric and the distortional component of $\boldsymbol{F}$ coincides with $\boldsymbol{F}$. A well-known example is the simple shear of a two-dimensional block as defined by the motion

$$x_1 = X_1 + \gamma X_2,$$
$$x_2 = X_2,$$

for any arbitrary value of γ. A simple derivation gives the deformation gradient and its Jacobean J as

$$\boldsymbol{F} = \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix}; \quad J = \det \boldsymbol{F} = 1;$$

and the Lagrangian and Eulerian deformation tensors are

$$\boldsymbol{E} = \frac{1}{2}\begin{bmatrix} 0 & \gamma \\ \gamma & \gamma^2 \end{bmatrix}; \quad \boldsymbol{e} = \frac{1}{2}\begin{bmatrix} 0 & \gamma \\ \gamma & -\gamma^2 \end{bmatrix}.$$

Consider an element of area in the initial configuration $d\boldsymbol{A} = dA\,\boldsymbol{N}$ which after deformation becomes $d\boldsymbol{a} = da\,\boldsymbol{n}$ as shown in Figure 4.7. For the purpose of obtaining a relationship between these two vectors, consider an arbitrary material vector $d\boldsymbol{L}$, which after deformation pushes forward to $d\boldsymbol{l}$. The corresponding initial and current volume elements are

$$dV = d\boldsymbol{L} \cdot d\boldsymbol{A}; \tag{4.66a}$$

$$dv = d\boldsymbol{l} \cdot d\boldsymbol{a}. \tag{4.66b}$$

Relating the current and initial volumes in terms of the Jacobian J and recalling that $d\boldsymbol{l} = \boldsymbol{F}\,d\boldsymbol{L}$ gives

$$J d\boldsymbol{L} \cdot d\boldsymbol{A} = (\boldsymbol{F} d\boldsymbol{L}) \cdot d\boldsymbol{a}. \tag{4.67}$$

The fact that the above expression is valid for any vector $d\boldsymbol{L}$ enables the elements of area to be related as

$$d\boldsymbol{a} = J\boldsymbol{F}^{-T} d\boldsymbol{A}. \tag{4.68}$$

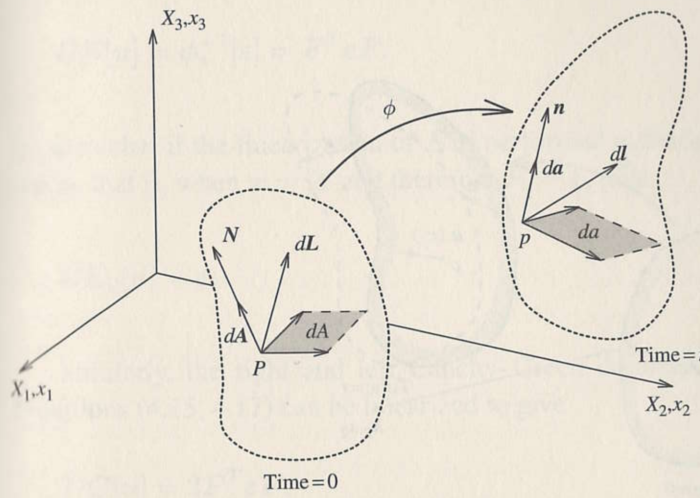


FIGURE 4.7 Area change.