PROGRAM ROZWOJOWY
POLITECHNIKI WARSZAWSKIEJ

LECTURE 8

NAVIER-STOKES EQUATION

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This lecture begins with derivation of the **equation of motion of Newtonian fluids**. Earlier, we have derived the general form from the 2nd Principle of Newton's dynamics

$$\rho \frac{D\upsilon_i}{Dt} \equiv \rho \left(\frac{\partial \upsilon_i}{\partial t} + \upsilon_j \frac{\partial \upsilon_i}{\partial x_j} \right) = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho\, f_i$$

Let us recall that the constitutive relation for Newtonian fluids reads

$$\sigma_{ij} = \left[-p + (\zeta - \tfrac{2}{3}\mu) \frac{\partial \upsilon_k}{\partial x_k} \right] \delta_{ij} + \mu \left[\frac{\partial \upsilon_i}{\partial x_j} + \frac{\partial \upsilon_j}{\partial x_i} \right]$$

We have to calculate the explicit form of the first term in the right-hand side of the equation of motion:

$$\frac{\partial \sigma_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_j}\delta_{ij} + (\zeta - \tfrac{2}{3}\mu)\frac{\partial}{\partial x_j}\left(\frac{\partial \upsilon_k}{\partial x_k}\right)\delta_{ij} + \mu\frac{\partial}{\partial x_j}\left(\frac{\partial \upsilon_i}{\partial x_j} + \frac{\partial \upsilon_j}{\partial x_i}\right) =$$

$$= -\frac{\partial p}{\partial x_i} + (\zeta - \tfrac{2}{3}\mu)\frac{\partial}{\partial x_i}\left(\frac{\partial \upsilon_k}{\partial x_k}\right) + \mu\frac{\partial^2 \upsilon_i}{\partial x_j \partial x_j} + \mu\frac{\partial}{\partial x_i}\left(\frac{\partial \upsilon_j}{\partial x_j}\right) =$$

$$= -\frac{\partial p}{\partial x_i} + (\zeta + \tfrac{1}{3}\mu)\frac{\partial}{\partial x_i}\left(\frac{\partial \upsilon_k}{\partial x_k}\right) + \mu\frac{\partial^2 \upsilon_i}{\partial x_j \partial x_j}$$

After the obtained formula is inserted into the equation of motion, we get

$$\rho\left(\frac{\partial \upsilon_i}{\partial t} + \upsilon_j\frac{\partial \upsilon_j}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + (\zeta + \tfrac{1}{3}\mu)\frac{\partial}{\partial x_i}\left(\frac{\partial \upsilon_k}{\partial x_k}\right) + \mu\frac{\partial^2 \upsilon_i}{\partial x_j \partial x_j} + \rho f_i$$

In the frame-invariant form, our equation of motion reads

$$\boxed{\rho\left[\frac{\partial \boldsymbol{\upsilon}}{\partial t} + (\boldsymbol{\upsilon}\cdot\nabla)\boldsymbol{\upsilon}\right] = -\nabla p + \mu\Delta\boldsymbol{\upsilon} + (\zeta + \tfrac{1}{3}\mu)\nabla(\nabla\cdot\boldsymbol{\upsilon}) + \rho\boldsymbol{f}}$$

This is the **Navier-Stokes Equation (NSE)**, the central equation of the dynamics of Newtonian fluids.

For an **incompressible fluid** $\nabla \cdot \boldsymbol{v} = 0$, so the **NSE** simplifies to

$$\rho \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla) \boldsymbol{v} \right] = -\nabla p + \mu \Delta \boldsymbol{v} + \rho \boldsymbol{f} \, ,$$

often also written in the form

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla) \boldsymbol{v} = -\frac{1}{\rho} \nabla p + \nu \Delta \boldsymbol{v} + \boldsymbol{f}$$

where $\nu = \mu / \rho$ is the **kinematic viscosity** of fluid (the SI unit is m^2/s).

The **index form** of the “incompressible” **NSE** is

$$\frac{\partial \upsilon_i}{\partial t} + \upsilon_j \frac{\partial \upsilon_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 \upsilon_i}{\partial x_j \partial x_j} + f_i$$

The Navier-Stokes Equation is the **vector equation** (or three scalar equations) with **four unknown fields**:

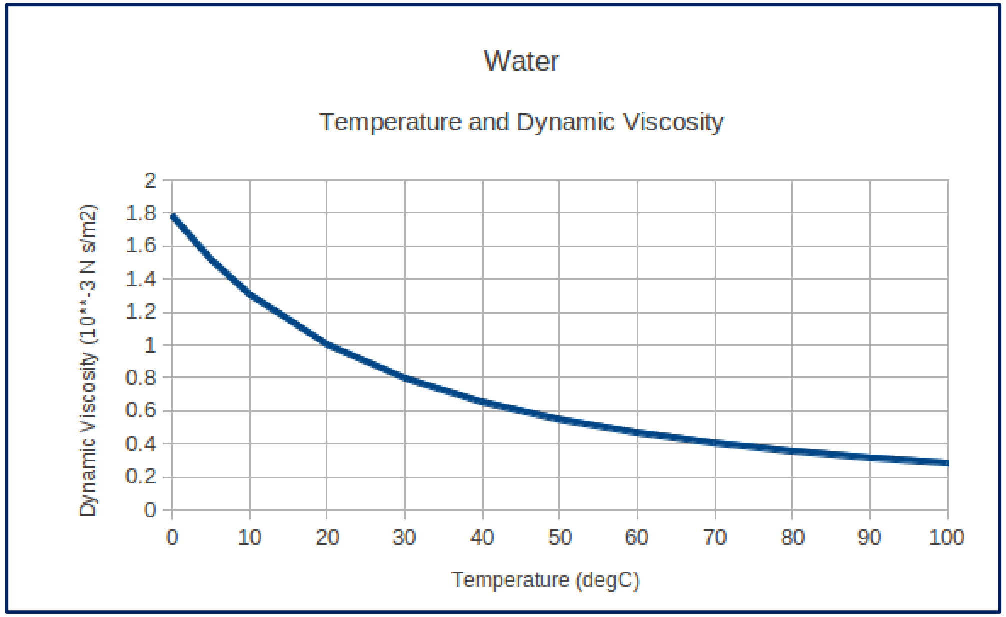
- three Cartesian components of the **velocity** field and
- the **pressure** field.

For an **incompressible fluid** it is sufficient to add the **continuity equation** $\nabla \cdot \boldsymbol{v} = 0$ and appropriate **initial and boundary conditions** to obtain a solvable mathematical problem. **However, "solvable" does not mean "easy to solve"!**

On the other hand, we need more equations when the **fluid is compressible**, since we have **one more unknown** – the **density** ρ. This additional equation is the scalar **equation of (total) energy conservation**. We will derive this equations in one of the next lectures.

Additional complication comes from the fact that **viscosity is temperature dependent**!

As a rule, viscosity of liquids diminishes with rising temperature.



For gases, the tendency is opposite. Typically, one can use the Sutherland formula

$$\mu = C \frac{T\sqrt{T}}{T+S}$$

(for air $S \approx 110K$, $C \approx 1.5 \cdot 10^{-6}$ ...)

We will discuss shortly the problem of boundary conditions for the Navier-Stokes Equations. We skip discussion of the compressible case, leaving this issue to more advanced courses.

In general, we have several kinds of the boundaries of the flow domain:

- **solid boundaries**: surfaces when the fluid is in contact with solid (rigid or elastic) walls
- **inflows and outflows**: surfaces through which stream of fluid enters or leaves the flow domain; such (artificial) boundaries are typical more modeling internal flows
- **far-field boundaries**: surfaces which are artificially introduced to bound a flow region around an immersed body (like an airplane) to the finite subset in space; such boundaries are typical for external flows

For liquid we may also have **free-surface conditions** (interface between liquid and ambient atmosphere).

The **boundary condition** at solid and impermeable surfaces (of the immersed bodies) is formulated as

$\boldsymbol{v} = \boldsymbol{u}$ at $\partial\Omega$, $\boldsymbol{u}$ - velocity of the boundary points

The **physical meaning** of the this conditions is that **viscous fluid adheres to a solid surface**, i.e. the velocities of the fluid and of the surface **are equal** (the **no-slip condition).**

What concerns the inlet/outlet conditions, we have a whole repertoire of possibilities. For instance, at the **inlets** one can prescribe the whole velocity vector or just its normal component plus distribution of the tangent component of the stress vector. At the **outlet** sections one can again prescribe the pressure and also assume that the tangent component of velocity is zero. Other options – better or worse suited for a particular physical situation – exist. However, **some combinations are not allowed**. For instance, **it is incorrect to impose simultaneously inlet/outlet distribution of the normal velocity and normal stress** (or pressure).

What concerns the far field, the boundary conditions are imposed to approximate the exact condition

$$\lim_{|\boldsymbol{x}|\to\infty} \boldsymbol{v} = \boldsymbol{v}_\infty$$

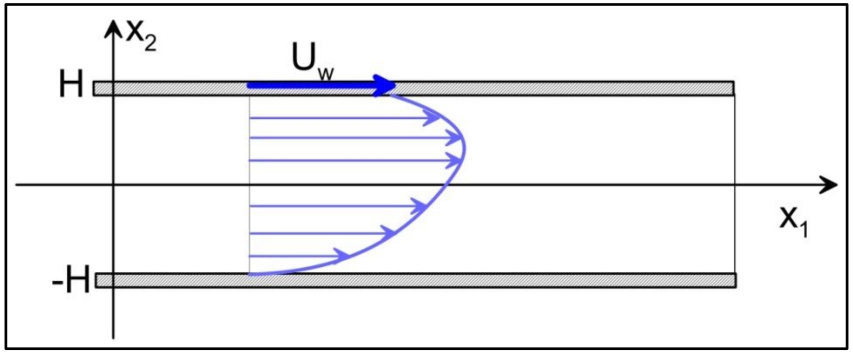
The simplest (but not the best) idea to is to “shift” this condition to the outer boundary of the finite fluid domain. Better approach relies on the idea of matching “internal” solution of the full NSE with some “outer” solution of some simplified flow model. Details of such approach are usually problem-dependent, thus we will not go into details.

ANALYTICAL SOLUTIONS OF NSE

It natural to ask if any **analytical solutions** to NSE exist. The answer is positive, however only few of them can be found using elementary techniques.

The standard examples of the analytical solutions to NSE include: Poiseuille-Couette flow in the plane (2D) channel, flow in the straight duct (in particular with circular or elliptic section), plane flow between two coaxial cylinders (Taylor-Couette flow) as well as few examples of time-dependent flows.

Example 1: Poiseuille-Couette flows



Flow is driven by movement of the upper wall (velocity of the wall is purely horizontal and equal U_W) and given pressure gradient.

The velocity field has only streamwise component

$$\boldsymbol{\upsilon} = [\upsilon_1, \upsilon_2] = [\upsilon_1, 0] \quad , \quad \upsilon_1 = \upsilon_1(x_2) \ , \quad \upsilon_2 \equiv 0 \quad , \qquad \frac{\partial}{\partial x_1}\upsilon_1 = 0$$

Continuity equation is satisfied automatically. The equations of motion reduce to very simple form

$$\begin{cases} 0 = -\frac{\partial}{\partial x_1} p + \mu \frac{\partial^2}{\partial x_2^2} \upsilon_1 \\ 0 = -\frac{\partial}{\partial x_2} p \end{cases}$$

Thus, pressure changes only in the flow direction. In the first equations, each term must be equal to a constant (**they depend on different spatial coordinates**).

We have

$$\frac{\partial}{\partial x_1} p = -K = const \quad (K > 0 \text{ is given})$$

The velocity can be found as follows

$$\frac{d^2}{d x_2^2} \upsilon_1 = -\frac{1}{\mu} K \;\Rightarrow\; \upsilon_1(x_2) = -\frac{1}{2\mu} K x_2^2 + A x_2 + B$$

The integration constants A and B are to be determined using the boundary conditions

$$x_2 = -H \quad \Rightarrow \upsilon_1(x_2) = 0$$

$$x_2 = H \quad \Rightarrow \upsilon_1(x_2) = U_w$$

After simple algebra we arrive at the final solution

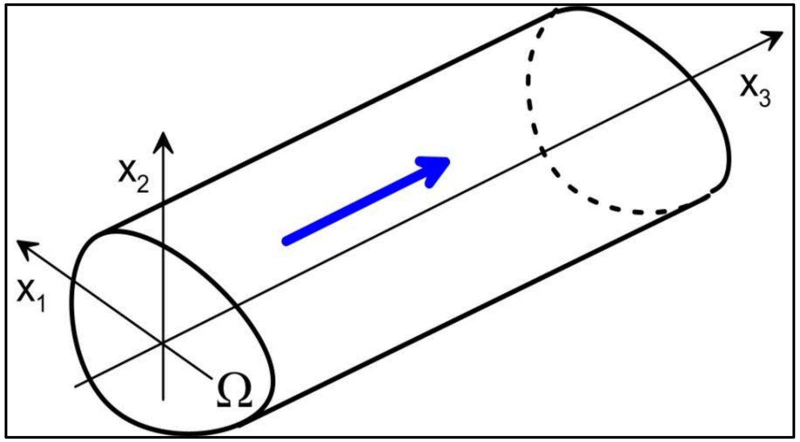
$$\upsilon_1(x_2) = \frac{1}{2}\frac{K}{\mu}\left[1 - \left(\frac{x_2}{H}\right)^2\right] + \frac{U_w}{2H}(x_2 + H)$$

We have two special cases:

$$U_w = 0 \quad \Rightarrow \; Poiseuille\ flow:\ \upsilon_1(x_2) = \frac{1}{2}\frac{K}{\mu}\left[1 - \left(\frac{x_2}{H}\right)^2\right]$$

$$K = 0 \quad \Rightarrow \; Couette\ flow: \quad \upsilon_1(x_2) = \frac{U_w}{2H}(x_2 + H)$$

Example 2: unidirectional flow in the duct with constant section



Flow is driven by the streamwise pressure gradient. There exist one nonzero component of the velocity field:

$$\boldsymbol{\upsilon} = [\upsilon_1, \upsilon_2, \upsilon_3] = [0, 0, \upsilon_3] \quad , \quad \upsilon_3 = \upsilon_3(x_1, x_2)$$

Similar argument as before leads to the conclusion that pressure depends only on x_3 and the pressure gradient is constant along the duct:

$$\frac{\partial}{\partial x_3} p = -K = const \quad (K > 0 \text{ is given})$$

The equation of motion becomes again very simple (Poisson equation)

$$\begin{cases} \frac{\partial^2}{\partial x_1^2}\upsilon_3 + \frac{\partial^2}{\partial x_2^2}\upsilon_3 = -\frac{K}{\mu} \quad in \;\; \Omega \\ \upsilon_3\big|_{\partial\Omega} = 0 \end{cases}$$

where Ω denotes the section of the duct.

The solution to the above boundary value problem may be found in the analytic form for a number of shapes. In general, the approximate solution can be found using appropriate numerical methods.

Special case: circular pipe

It is natural to use cylindrical polar coordinate system. Assuming that the flow field is axisymmetric, the equation of motion reduces to the following ordinary differential equation (we use the symbol $w = \upsilon_3$)

$$\frac{d^2}{dr^2} w + \frac{1}{r}\frac{d}{dr} w \equiv \frac{1}{r}\left[\frac{d}{dr}\left(r\frac{d}{dr} w\right)\right] = -\frac{K}{\mu}$$

The boundary conditions are:

$$\frac{d}{dr} w(r=0) = 0 \quad , \quad w(r=R) = 0$$

The solution to the above boundary value problem can be found in the following form

$$w(r) = \underbrace{\frac{KR^2}{4\mu}}_{w_0}\left[1-\left(\frac{r}{R}\right)^2\right] = w_0\left[1-\left(\frac{r}{R}\right)^2\right]$$

This is the **Hagen-Poiseuille flow**.

Let us compute the volumetric flow rate of thus flow:

$$Q = \int_{\Omega} w dS = 2\pi \int_0^R w(r) r dr = \frac{K}{8\mu} \pi R^4 = K \frac{\pi D^4}{128\mu}$$

This is the **Hagen-Poiseuille formula**. Note that the flow rate is proportional to the pressure gradient and inverse proportional to fluid viscosity.

We will show that some dimensionless measure of the flow resistance can be defined. To this aim let us calculate the average velocity

$$w_{av} = \frac{Q}{\frac{1}{4}\pi D^2} = \frac{KD^2}{32\mu} = \frac{1}{2} w_0$$

Then, the pressure gradient needed to sustain the flow rate Q can be recalculated into dimensionless coefficient of distributed pressure losses λ:

$$K = \frac{32 w_{av} \rho \nu}{D^2} \Rightarrow \lambda \equiv \frac{KD}{\frac{1}{2}\rho w_{av}^2} = \frac{64}{\frac{w_{av} D}{\nu}} = \frac{64}{\mathrm{Re}} \quad , \quad \mathrm{Re} = \frac{w_{av} D}{\nu}$$

In the above, we have introduces a very important dimensionless quantity – the **Reynolds number** Re. We will say more about this number in the lecture about dynamic similitude of flows.

Other cases with analytical solution include: the elliptic pipe, the pipe with equilateral triangular section and the pipe with rectangular section (for the latter, formulas have the form of the infinite series).

Also, analytical solutions exist for a few simple nonstationary flows (e.g., Womersley flow, i.e., flow in the pipe driven by an oscillatory pressure gradient).

APPENDIX

The Navier-Stokes Equations (incompressible flow) in cylindrical/polar coordinates (R, θ, z)

$$\rho\left[\partial_t \upsilon_R + (\boldsymbol{\upsilon}\cdot\nabla)\upsilon_R - \frac{\upsilon_\theta^2}{R}\right] = -\partial_R p + \mu\left[\nabla^2\upsilon_R - \frac{\upsilon_R}{R^2} - \frac{2}{R^2}\partial_\theta \upsilon_\theta\right]$$

$$\rho\left[\partial_t \upsilon_\theta + (\boldsymbol{\upsilon}\cdot\nabla)\upsilon_\theta + \frac{\upsilon_R \upsilon_\theta}{R}\right] = -\frac{1}{R}\partial_\theta p + \mu\left[\nabla^2\upsilon_\theta - \frac{\upsilon_\theta}{R^2} + \frac{2}{R^2}\partial_\theta \upsilon_R\right]$$

$$\rho\left[\partial_t \upsilon_z + (\boldsymbol{\upsilon}\cdot\nabla)\upsilon_z\right] = -\partial_z p + \mu\nabla^2\upsilon_z$$

Where $\boldsymbol{\upsilon}\cdot\nabla = \upsilon_R \partial_R + \frac{1}{R}\upsilon_\theta \partial_\theta + \upsilon_z \partial_z$, $\nabla^2 = \frac{1}{R}\partial_R(R\partial_R) + \frac{1}{R^2}\partial_{\theta\theta} + \partial_{zz}$

Continuity equation: $\frac{1}{R}\partial_R(R\upsilon_R) + \frac{1}{R}\partial_\theta \upsilon_\theta + \partial_z \upsilon_z = 0$

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