Warsaw University of Technology

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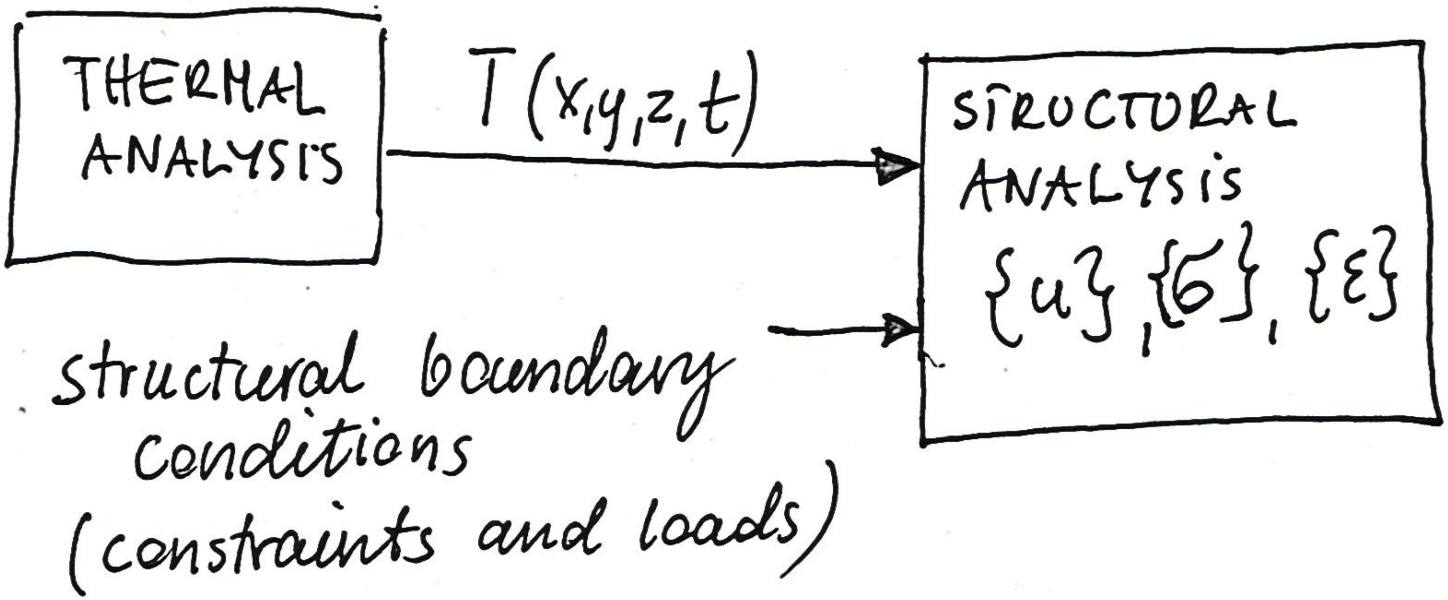
WARSAW UNIVERSITY OF TECHNOLOGY

Institute of Aeronautics and Applied Mechanics

Finite element method 2 (FEM 2)

Structural analysis with thermal effect

THERMAL STRESS ANALYSIS



STRUCTURAL ANALYSIS WITH THERMAL EFFECT

TOTAL STRAIN:

$$\underset{6\times1}{\{\varepsilon\}} = \underset{6\times1}{\{\varepsilon\}_T} + \underset{6\times1}{\{\varepsilon\}_e}$$

$$\underset{6\times1}{\{\varepsilon\}} = \underset{6\times n_e}{[B]} \cdot \underset{n_e\times1}{\{q\}_e}$$

n_e - no. of degrees of freedom in a finite element

STRESS

$$\underset{6\times1}{\{\sigma\}} = \underset{6\times6}{[D]} \cdot \underset{6\times1}{\{\varepsilon\}_e} = \underset{6\times6}{[D]} \cdot \left(\underset{6\times1}{\{\varepsilon\}} - \underset{6\times1}{\{\varepsilon\}_T} \right) =$$

constitutive matrix

$$= \underset{6\times6}{[D]} \left(\underset{6\times n_e}{[B]} \cdot \underset{n_e\times1}{\{q\}_e} - \underset{6\times1}{\{\varepsilon\}_T} \right)$$

Virtual work principle:

$$\underset{1\times n_e}{\lfloor \delta q \rfloor_e} \cdot \underset{n_e\times 1}{\{F\}_e} = \int_{\Omega_e} \underset{1\times 6}{\lfloor \delta \varepsilon \rfloor} \cdot \underset{6\times 1}{\{\sigma\}} \, d\Omega_e$$

virtual displacement nodal forces virtual strain

$$\underset{1\times n_e}{\lfloor \delta q \rfloor_e} \cdot \underset{n_e\times 1}{\{F\}_e} - \int_{\Omega_e} \underset{1\times 6}{\lfloor \delta \varepsilon \rfloor} \cdot \underset{6\times 1}{\{\sigma\}} \, d\Omega_e = 0$$

$$\underset{1\times 6}{\lfloor \delta \varepsilon \rfloor} = \underset{1\times n_e}{\lfloor \delta q \rfloor_e} \cdot \underset{n_e\times 6}{[B]^T}$$

hence:

$$\underset{1\times n_e}{\lfloor \delta q \rfloor_e} \underbrace{\left(\underset{n_e\times 1}{\{F\}_e} - \int_{\Omega_e} \underset{n_e\times 6}{[B]^T} \cdot \underset{6\times 1}{\{\sigma\}} \, d\Omega_e \right)}_{=0} = 0$$

$$\underset{n_e\times 1}{\{F\}_e} = \int_{\Omega_e} \underset{n_e\times 6}{[B]^T} \cdot \underset{6\times 1}{\{\sigma\}}\, d\Omega_e = \int_{\Omega_e} \underset{n_e\times 6}{[B]^T} \cdot \underset{6\times 6}{[D]} \left(\underset{6\times n_e}{[B]} \cdot \underset{n_e\times 1}{\{q\}_e} - \underset{6\times 1}{\{\varepsilon\}_T} \right) d\Omega_e$$

$$= \underbrace{\int_{\Omega_e} \underset{n_e\times 6}{[B]^T} \underset{6\times 6}{[D]} \cdot \underset{6\times n_e}{[B]}\, d\Omega_e}_{\underset{n_e\times n_e}{[K]_e}} \cdot \underset{n_e\times 1}{\{q\}_e} - \underbrace{\int_{\Omega_e} \underset{n_e\times 6}{[B]^T} \cdot \underset{6\times 6}{[D]} \cdot \underset{6\times 1}{\{\varepsilon\}_T}\, d\Omega_e}_{\underset{n_e\times 1}{\{F_T\}_e}}$$

local stiffness matrix

vecter of thermal load

Finally:
$$\underset{n_e \times n_e}{[k]_e} \cdot \underset{n_e \times 1}{\{q\}_e} = \underset{n_e \times 1}{\{F\}_e} + \underset{n_e \times 1}{\{F_T\}_e}$$

nodal forces due to temperature

Set of equations for the entire FE model:

$$\underset{NDOF \times NDOF}{[K]} \cdot \underset{NDOF \times 1}{\{q\}} = \underset{NDOF \times 1}{\{F\}} + \underset{NDOF \times 1}{\{F_T\}}$$

global stiffness matrix

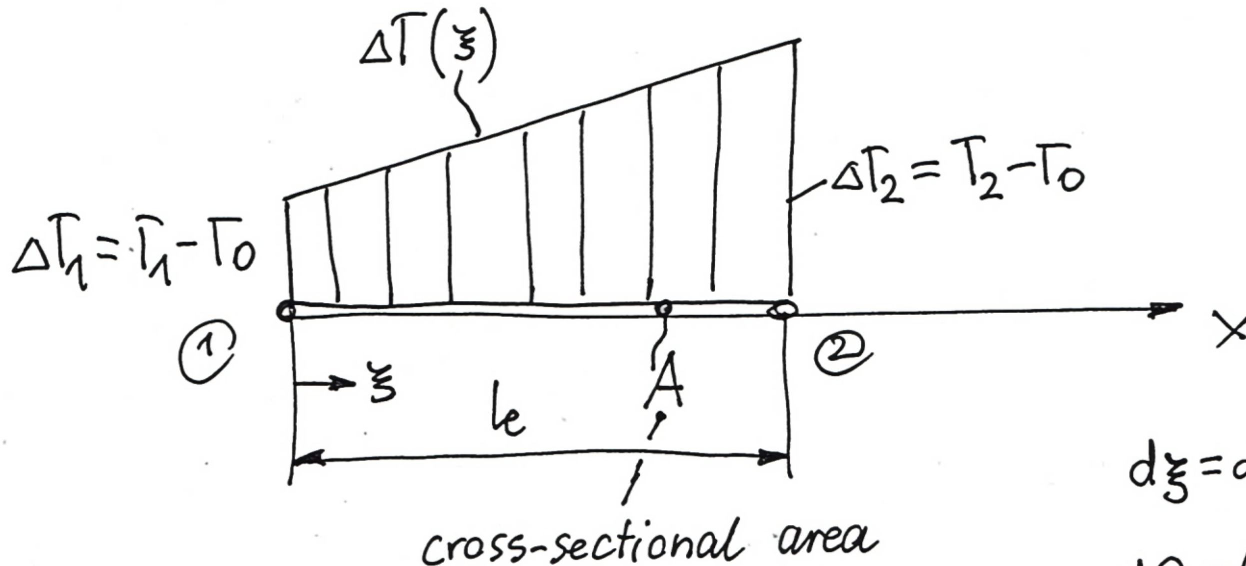
global vector of nodal parameters

global vector of structural load

global vector of thermal load

NDOF – number of degrees of freedom

EXAMPLE : FIND THERMAL LOAD VECTOR FOR A BAR FINITE ELEMENT.



$d\xi = dx$

$d\Omega_e = A \cdot d\xi$

$\Delta T_1, \Delta T_2$ - nodal parameters in thermal analysis

$$\Delta T(\xi) = \lfloor N_1(\xi), N_2(\xi) \rfloor \cdot \begin{Bmatrix} \Delta T_1 \\ \Delta T_2 \end{Bmatrix}$$

$$N_1(\xi) = 1 - \frac{\xi}{L_e} \quad , \quad N_2(\xi) = \frac{\xi}{L_e}$$

$$\varepsilon_T(\xi) = \alpha_{se} \cdot \Delta T(\xi) = \alpha_{se} \cdot \lfloor N_1, N_2 \rfloor \cdot \begin{Bmatrix} \Delta T_1 \\ \Delta T_2 \end{Bmatrix}$$

FOR „3D":

$$\underset{6 \times n_e}{[B]} = \underset{6 \times 3}{[R]} \cdot \underset{3 \times n_e}{[N]} \quad ;$$

FOR „1D":

$$\underset{1 \times 2}{\lfloor B \rfloor} = \frac{\partial}{\partial \xi} \lfloor N_1, N_2 \rfloor = \left\lfloor \frac{\partial N_1}{\partial \xi}, \frac{\partial N_2}{\partial \xi} \right\rfloor = \left\lfloor -\frac{1}{l_e}, \frac{1}{l_e} \right\rfloor$$

$$\underset{2 \times 1}{\{B\}} = \begin{Bmatrix} -\frac{1}{l_e} \\ \\ \frac{1}{l_e} \end{Bmatrix}$$

ASSUMING HOOKE'S LAW :

(FOR "3D" $\underset{6\times1}{\{\sigma\}} = \underset{6\times6}{[D]} \cdot \underset{6\times1}{\{\varepsilon\}_e}$) here: ("1D") $\sigma = E \cdot \varepsilon_e$ (along the bar axis)

THERMAL LOAD VECTOR :

$$\underset{2\times1}{\{F_T\}_e} = \int_{\Omega_e} \{B\} \cdot E \cdot \varepsilon_T(\xi)\, d\Omega_e =$$

$$= \int_0^{l_e} \begin{Bmatrix} -\frac{1}{l_e} \\ \frac{1}{l_e} \end{Bmatrix} E \cdot \alpha_{se} \cdot \lfloor N_1(\xi), N_2(\xi) \rfloor \cdot \begin{Bmatrix} \Delta T_1 \\ \Delta T_2 \end{Bmatrix} \cdot A \cdot d\xi =$$

$$= \alpha_{se} \cdot E \cdot A \cdot \int_0^{l_e} \begin{bmatrix} -\frac{N_1(\xi)}{l_e} & -\frac{N_2(\xi)}{l_e} \\ \frac{N_1(\xi)}{l_e} & \frac{N_2(\xi)}{l_e} \end{bmatrix} d\xi \cdot \begin{Bmatrix} \Delta T_1 \\ \Delta T_2 \end{Bmatrix} \Rightarrow$$

$$\Rightarrow \quad \underset{2\times 1}{\{F_T\}_e} = \alpha_{se} \cdot E \cdot A \cdot \int_0^{l_e} \begin{bmatrix} -\dfrac{\left(1-\frac{\xi}{l_e}\right)}{l_e} & -\dfrac{\xi}{l_e^2} \\ \dfrac{\left(1-\frac{\xi}{l_e}\right)}{l_e} & \dfrac{\xi}{l_e^2} \end{bmatrix} d\xi \cdot \begin{Bmatrix} \Delta T_1 \\ \Delta T_2 \end{Bmatrix} =$$

$$= \alpha_{se} \cdot E \cdot A \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \cdot \begin{Bmatrix} \Delta T_1 \\ \Delta T_2 \end{Bmatrix} =$$

$$= \frac{\Delta T_1 + \Delta T_2}{2} \alpha_{se} \cdot E \cdot A \cdot \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$