Warsaw University of Technology

Faculty of Power and Aeronautical Engineering
WARSAW UNIVERSITY OF TECHNOLOGY

Institute of Aeronautics and Applied Mechanics

Finite element method
(FEM)

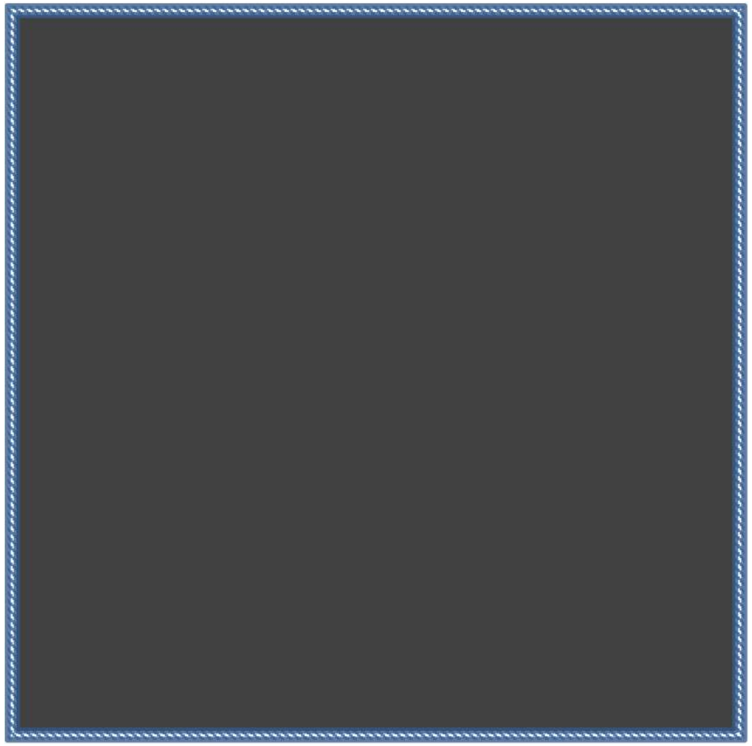
Finite Difference Method. A square plate

06.2021

Square plate

Young's modulus: $E = 2 \cdot 10^5$ *MPa*

thickness: $t = 2\ mm$

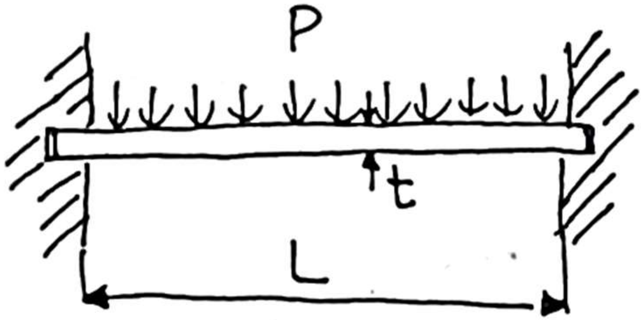


$L = 100\ mm$

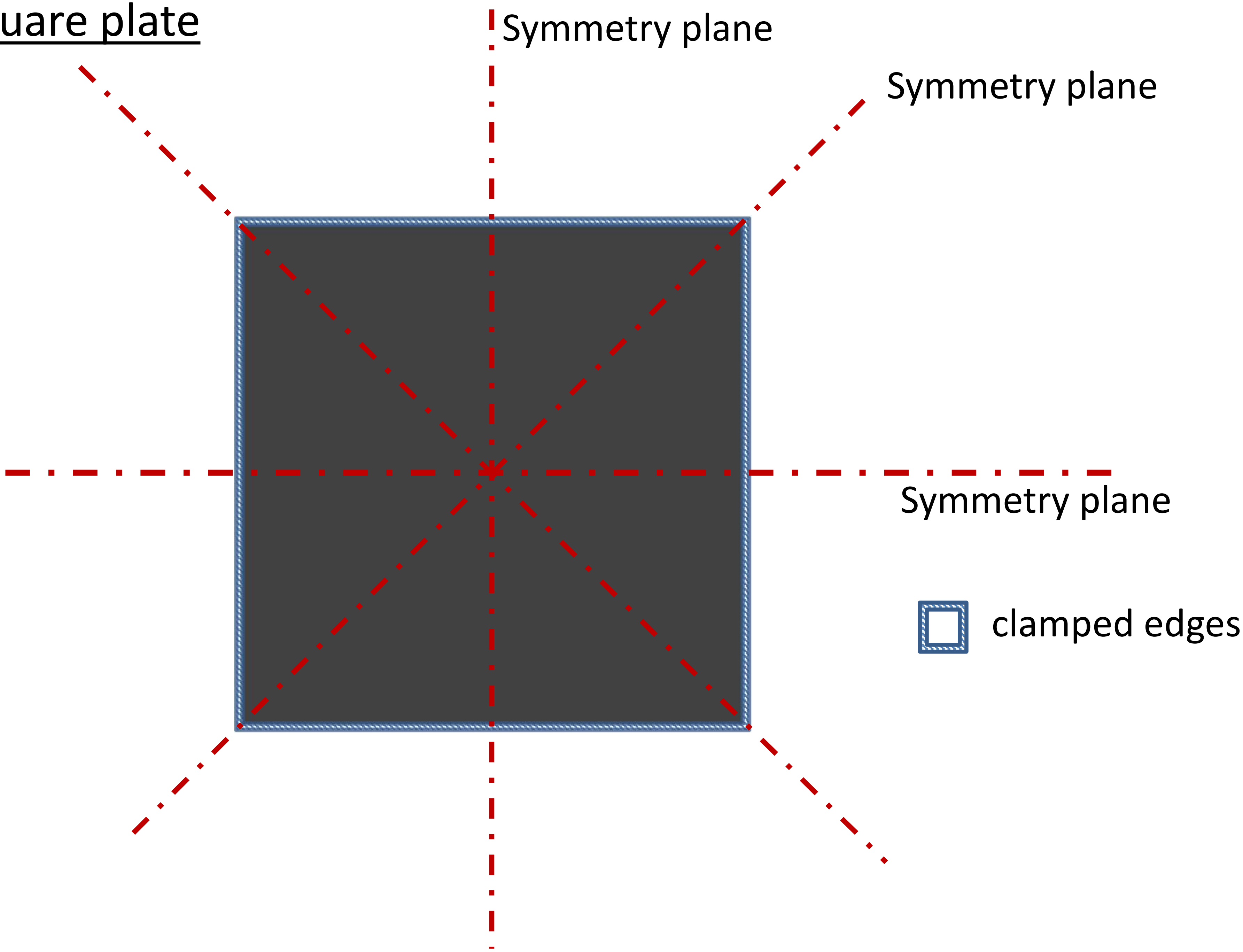
$L = 100\ mm$

uniform pressure: p=0.2 MPa

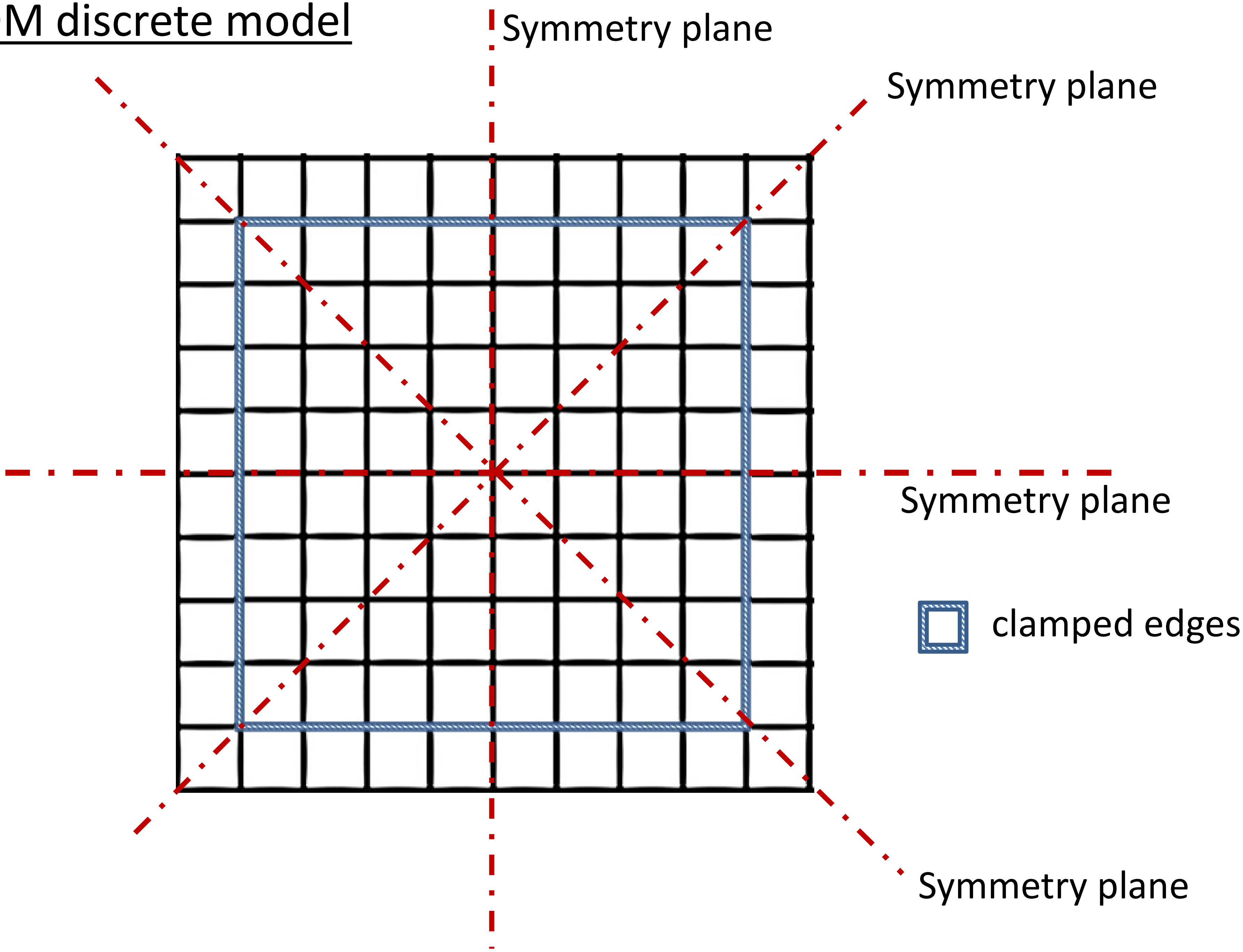
▣ clamped edges



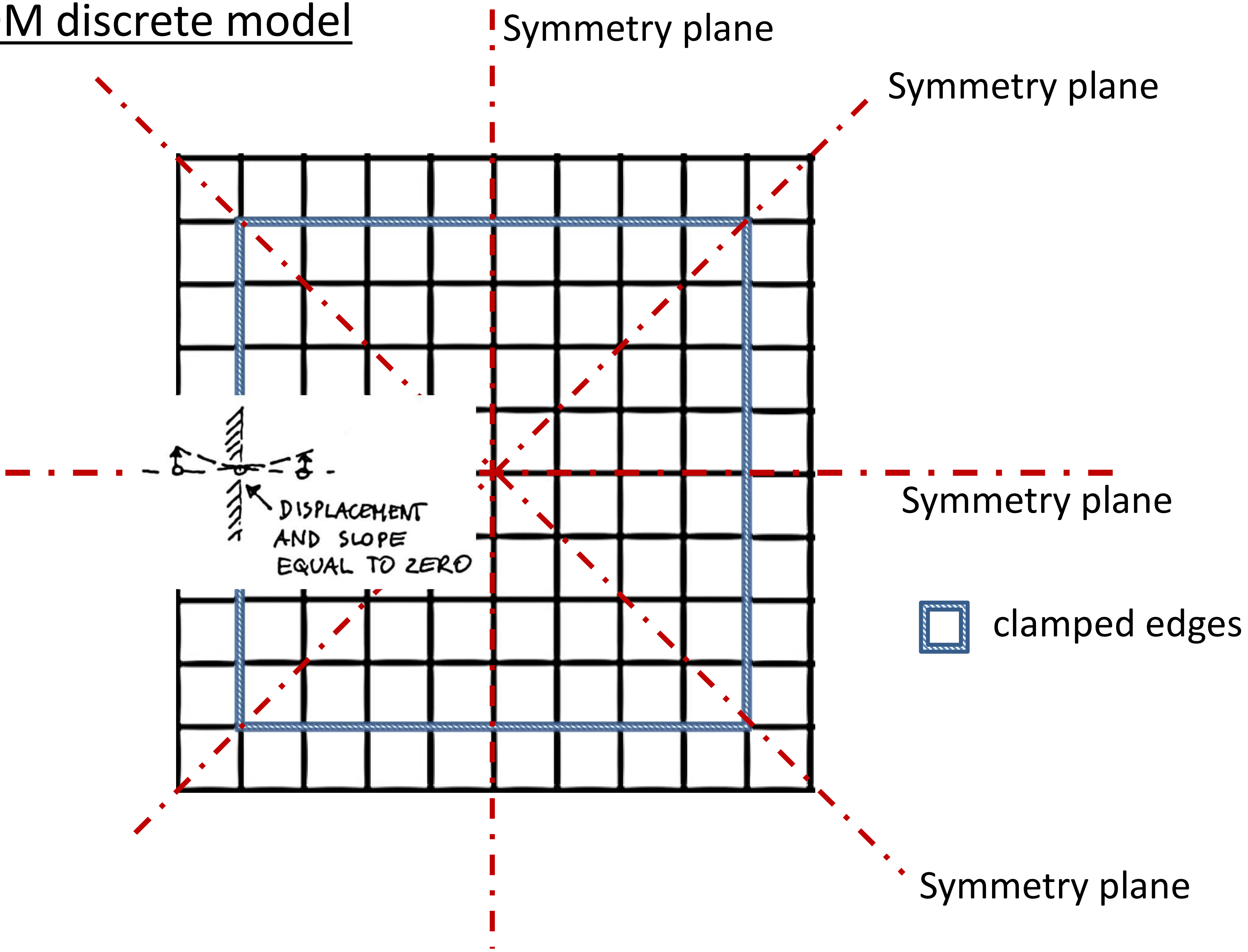
Square plate



FDM discrete model



FDM discrete model



Symmetry plane

Symmetry plane

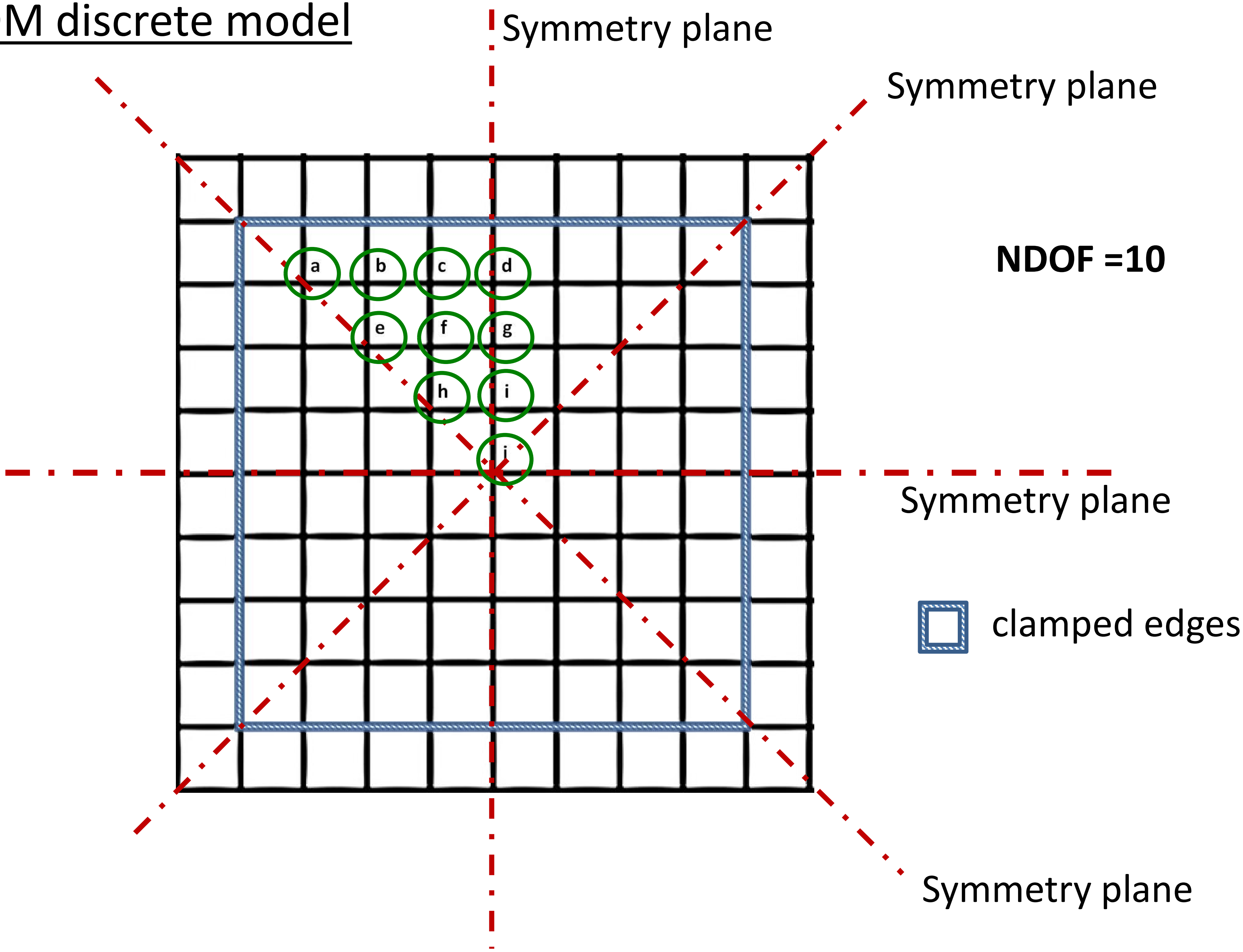
DISPLACEMENT AND SLOPE EQUAL TO ZERO

Symmetry plane

☐ clamped edges

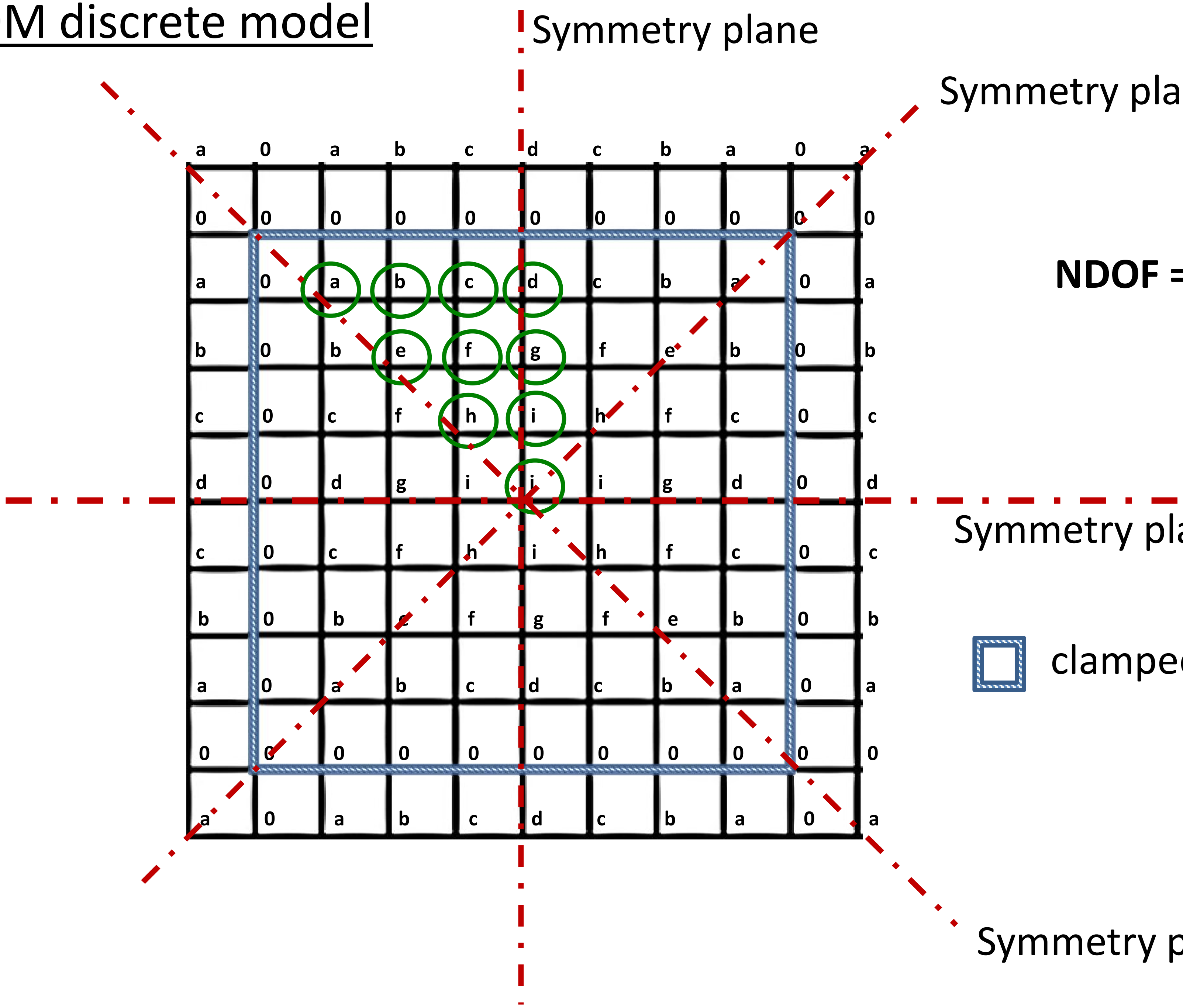
Symmetry plane

FDM discrete model



NDOF =10

FDM discrete model



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b	0	b	e	f	g	f	e	b	0	b
c	0	c	f	h	i	h	f	c	0	c
d	0	d	g	i	j	i	g	d	0	d
c	0	c	f	h	i	h	f	c	0	c
b	0	b	e	f	g	f	e	b	0	b
a	0	a	b	c	d	c	b	a	0	a
0	0	0	0	0	0	0	0	0	0	0
a	0	a	b	c	d	c	b	a	0	a

NDOF =10

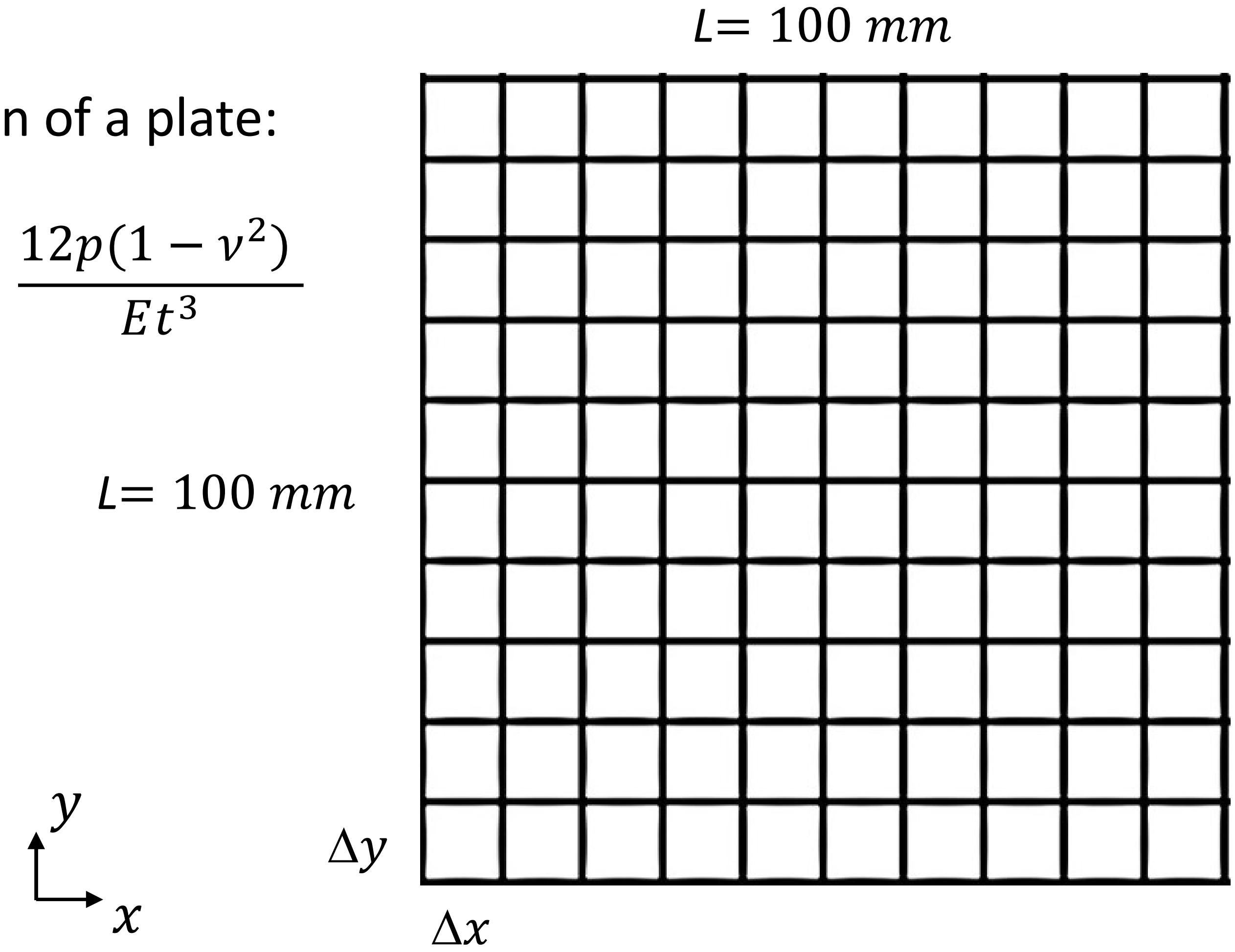
▢ clamped edges

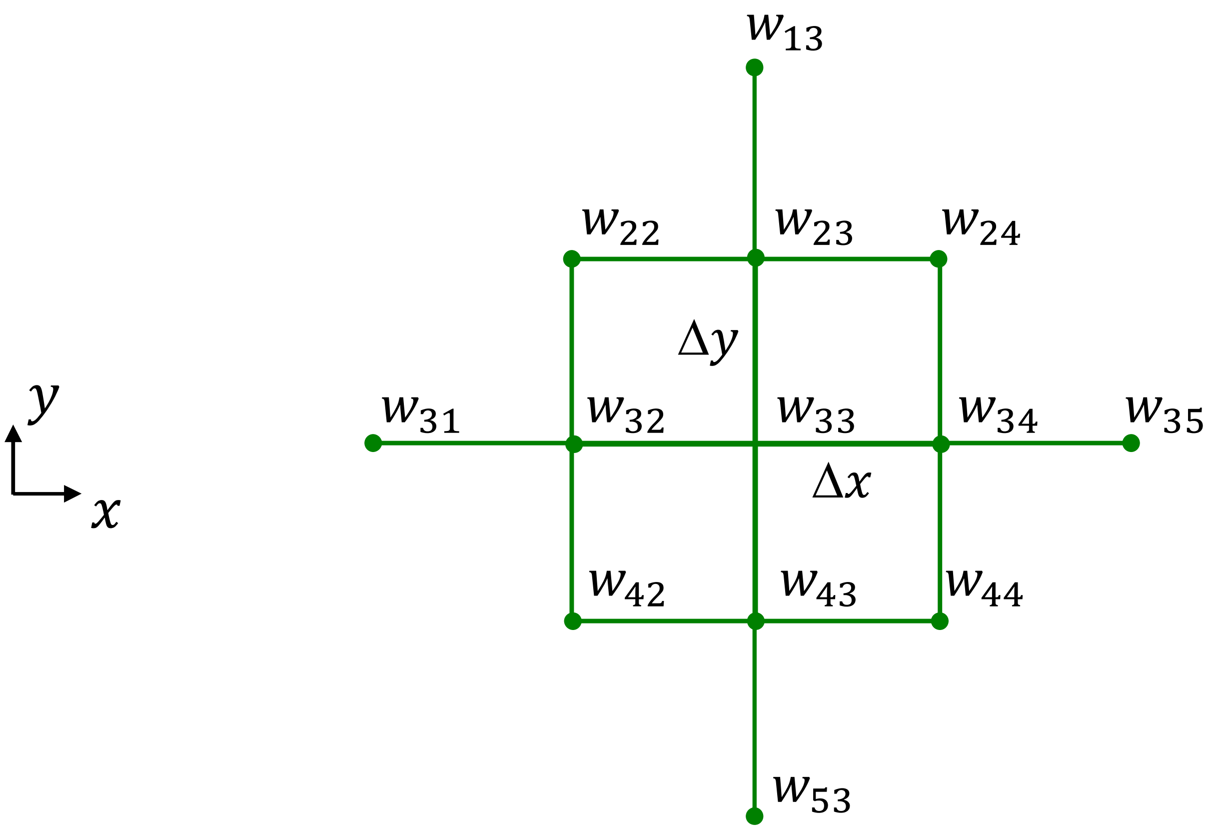
Partial differential equation of a plate:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{12p(1-\nu^2)}{Et^3}$$

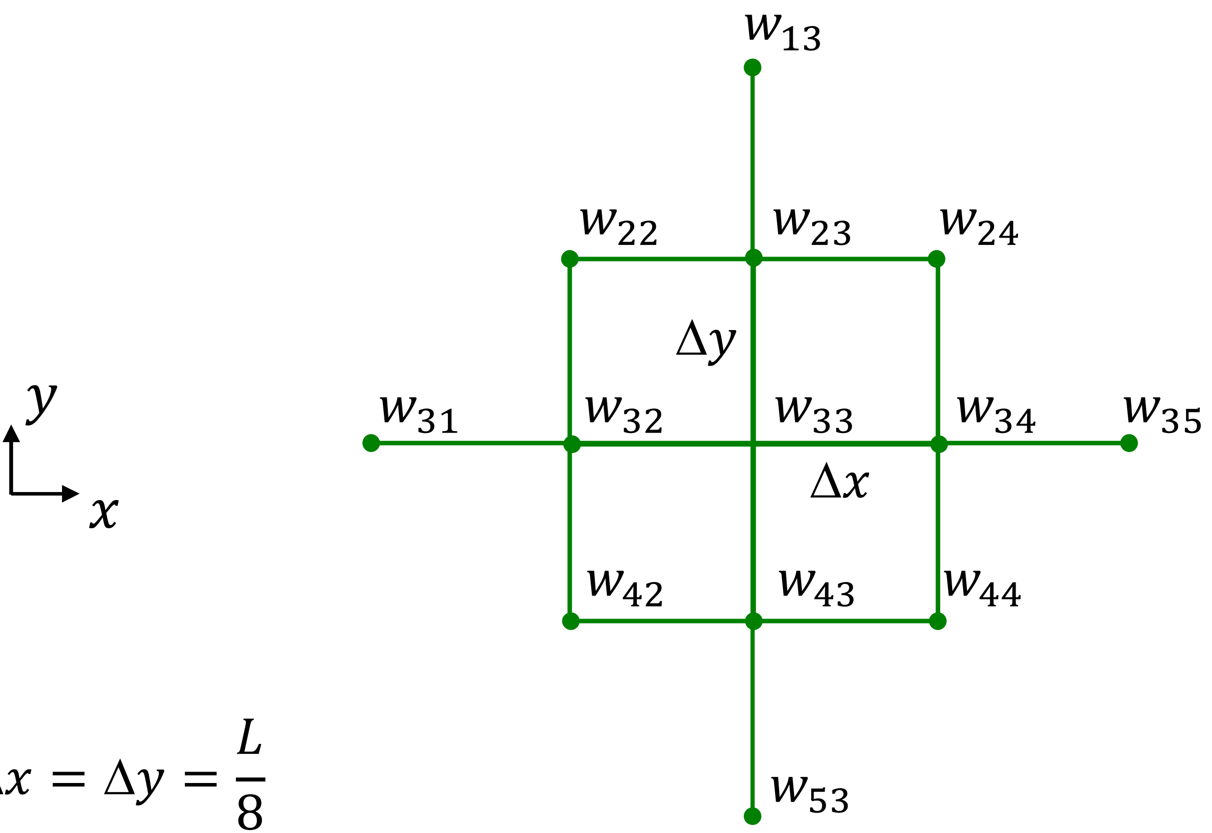
Finite difference equation of a plate:

$$\frac{\Delta^4 w}{\Delta x^4} + 2\frac{\Delta^4 w}{\Delta x^2 \Delta y^2} + \frac{\Delta^4 w}{\Delta y^4} = \frac{12p(1-\nu^2)}{Et^3}$$





$$\frac{1}{\Delta x^4}(w_{31} - 4w_{32} + 6w_{33} - 4w_{34} + w_{35}) +$$
$$+\frac{2}{\Delta x^2 \Delta y^2}(w_{22} - 2w_{23} + w_{24} - 2w_{32} + 4w_{33} - 2w_{34} + w_{42} - 2w_{43} + w_{44}) +$$
$$+\frac{1}{\Delta x^4}(w_{13} - 4w_{23} + 6w_{33} - 4w_{43} + w_{53}) = \frac{12p(1-\nu^2)}{Et^3}$$



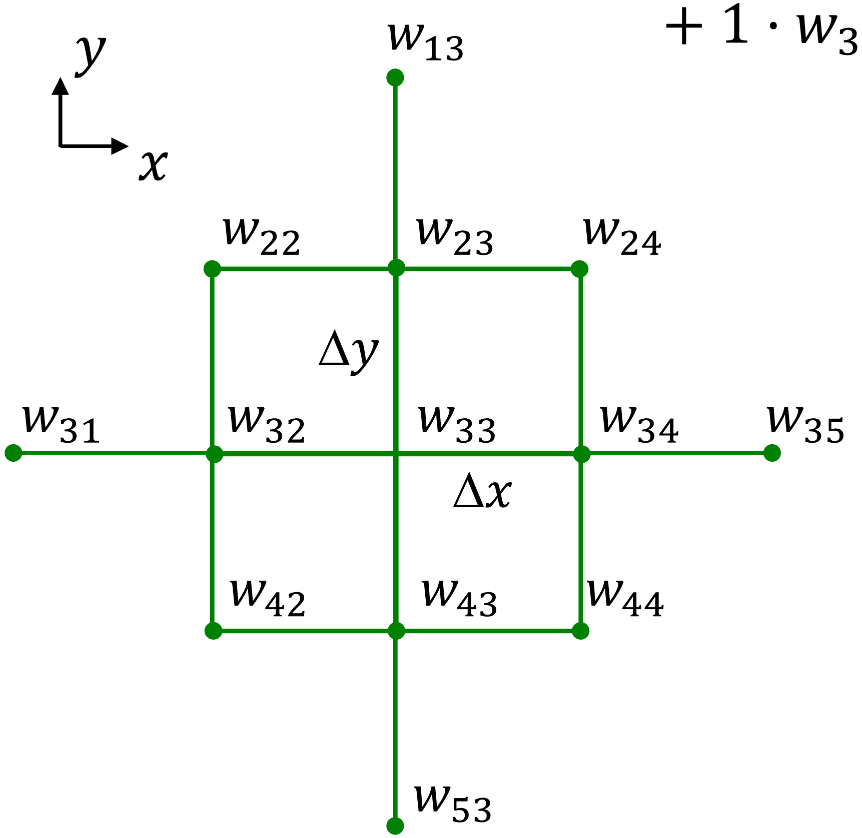
$$\Delta x = \Delta y = \frac{L}{8}$$

$$(w_{31} - 4w_{32} + 6w_{33} - 4w_{34} + w_{35}) +$$

$$+\,2(w_{22} - 2w_{23} + w_{24} - 2w_{32} + 4w_{33} - 2w_{34} + w_{42} - 2w_{43} + w_{44}) +$$

$$+(w_{13} - 4w_{23} + 6w_{33} - 4w_{43} + w_{53}) \;=$$

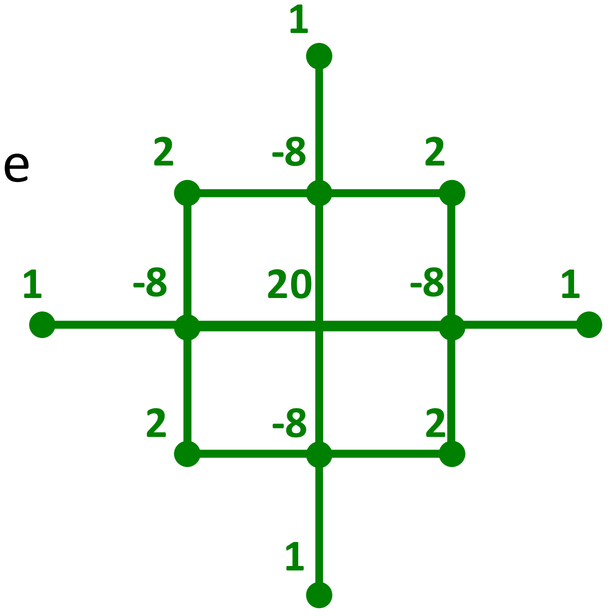
$$= \frac{12p(1-\nu^2)L^4}{4096Et^3}$$



$$
\begin{gathered}
1 \cdot w_{13} + \\
+ 2 \cdot w_{22} - 8 \cdot w_{23} + 2 \cdot w_{24} + \\
+ 1 \cdot w_{31} - 8 \cdot w_{32} + 20 \cdot w_{33} - 8 \cdot w_{34} + 1 \cdot w_{35} + \\
+ 2 \cdot w_{42} - 8 \cdot w_{43} + 2 \cdot w_{44} + \\
+ 1 \cdot w_{53}
\end{gathered}
$$

$$
= \frac{12p(1-\nu^2)L^4}{4096Et^3}
$$

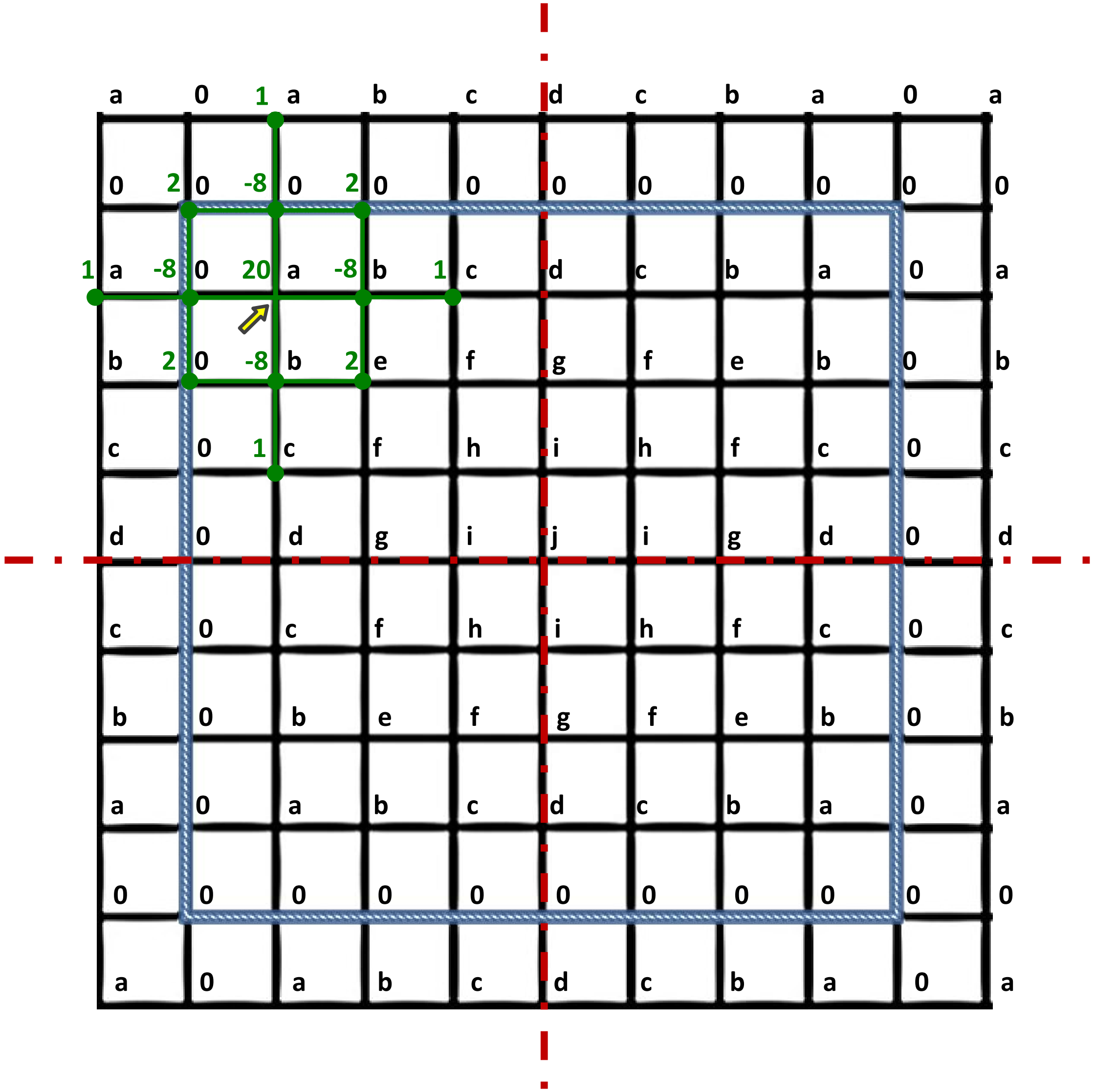
Differential scheme



$$
\Delta x = \Delta y = \frac{L}{8}
$$

$$1 \cdot a + 2 \cdot 0 - 8 \cdot 0 + 2 \cdot 0 + 1 \cdot a - 8 \cdot 0 + 20 \cdot a - 8 \cdot b + 1 \cdot c + 2 \cdot 0 - 8 \cdot b + 2 \cdot e + 1 \cdot c = \frac{12p(1-\nu^2)L^4}{4096Et^3}$$

$$22 \cdot a - 16 \cdot b + 2 \cdot c + 0 \cdot d + 2 \cdot e + 0 \cdot f + 0 \cdot g + 0 \cdot h + 0 \cdot i + 0 \cdot j = \frac{12p(1-\nu^2)L^4}{4096Et^3}$$

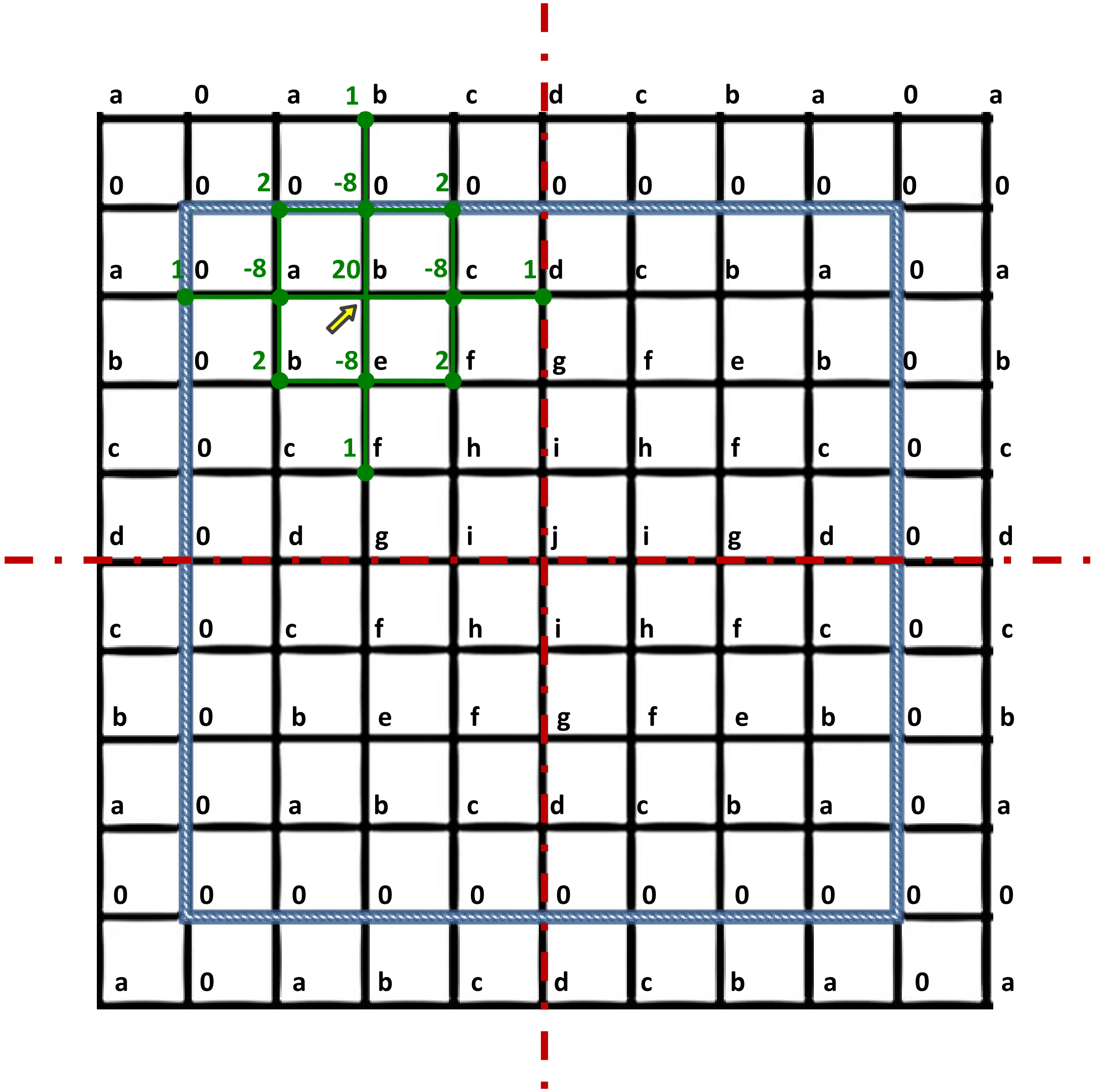


a	0	a	b	c	d	c	b	a	0	a
0	0	0	0	0	0	0	0	0	0	0
a	0	a	b	c	d	c	b	a	0	a
b	0	b	e	f	g	f	e	b	0	b
c	0	c	f	h	i	h	f	c	0	c
d	0	d	g	i	j	i	g	d	0	d
c	0	c	f	h	i	h	f	c	0	c
b	0	b	e	f	g	f	e	b	0	b
a	0	a	b	c	d	c	b	a	0	a
0	0	0	0	0	0	0	0	0	0	0
a	0	a	b	c	d	c	b	a	0	a

▢ clamped edges

$$1 \cdot b + 2 \cdot 0 - 8 \cdot 0 + 2 \cdot 0 + 1 \cdot 0 - 8 \cdot a + 20 \cdot b - 8 \cdot c + 1 \cdot d + 2 \cdot b - 8 \cdot e + 2 \cdot f + 1 \cdot f = \frac{12 \ \ (1-\nu^2)L^4}{4096Et^3}$$

$$-8 \cdot a + 23 \cdot b - 8 \cdot c + 1 \cdot d - 8 \cdot e + 3 \cdot f + 0 \cdot g + 0 \cdot h + 0 \cdot i + 0 \cdot j = \frac{12 \ \ (1-\nu^2)L^4}{4096Et^3}$$

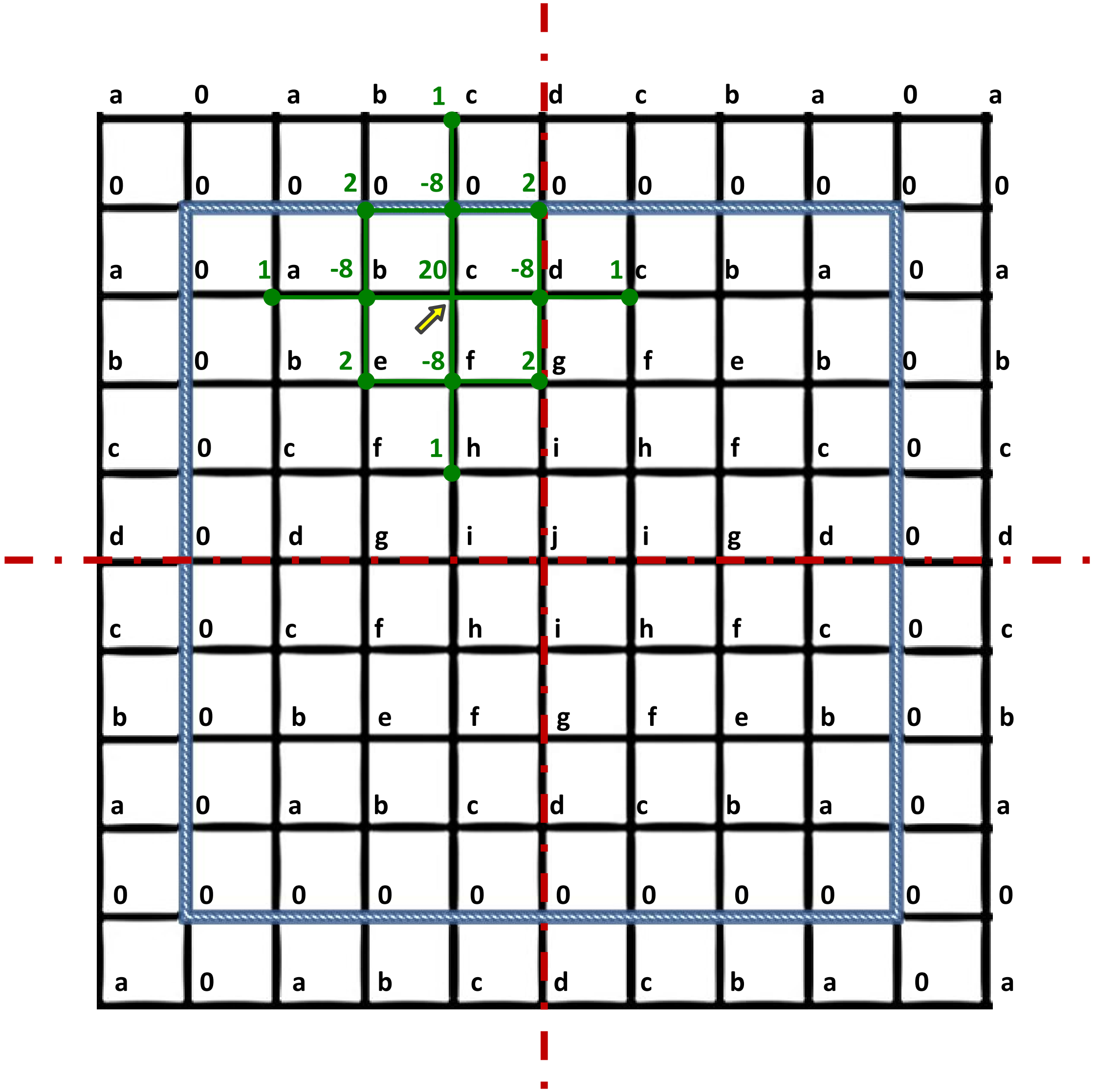


a	0	a	b	c	d	c	b	a	0	a
0	0	0	0	0	0	0	0	0	0	0
a	0	a	b	c	d	c	b	a	0	a
b	0	b	e	f	g	f	e	b	0	b
c	0	c	f	h	i	h	f	c	0	c
d	0	d	g	i	j	i	g	d	0	d
c	0	c	f	h	i	h	f	c	0	c
b	0	b	e	f	g	f	e	b	0	b
a	0	a	b	c	d	c	b	a	0	a
0	0	0	0	0	0	0	0	0	0	0
a	0	a	b	c	d	c	b	a	0	a

□ clamped edges

$$1\cdot c+2\cdot 0-8\cdot 0+2\cdot 0+1\cdot a-8\cdot b+20\cdot c-8\cdot d+1\cdot c+2\cdot e-8\cdot f+2\cdot g+1\cdot h=\frac{12p(1-\nu^2)L^4}{4096\quad {}^{3}}$$

$$1\cdot a-8\cdot b+22\cdot c-8\cdot d+2\cdot e-8\cdot f+2\cdot g+1\cdot h+0\cdot i+0\cdot j=\frac{12p(1-\nu^2)L^4}{4096\quad {}^{3}}$$

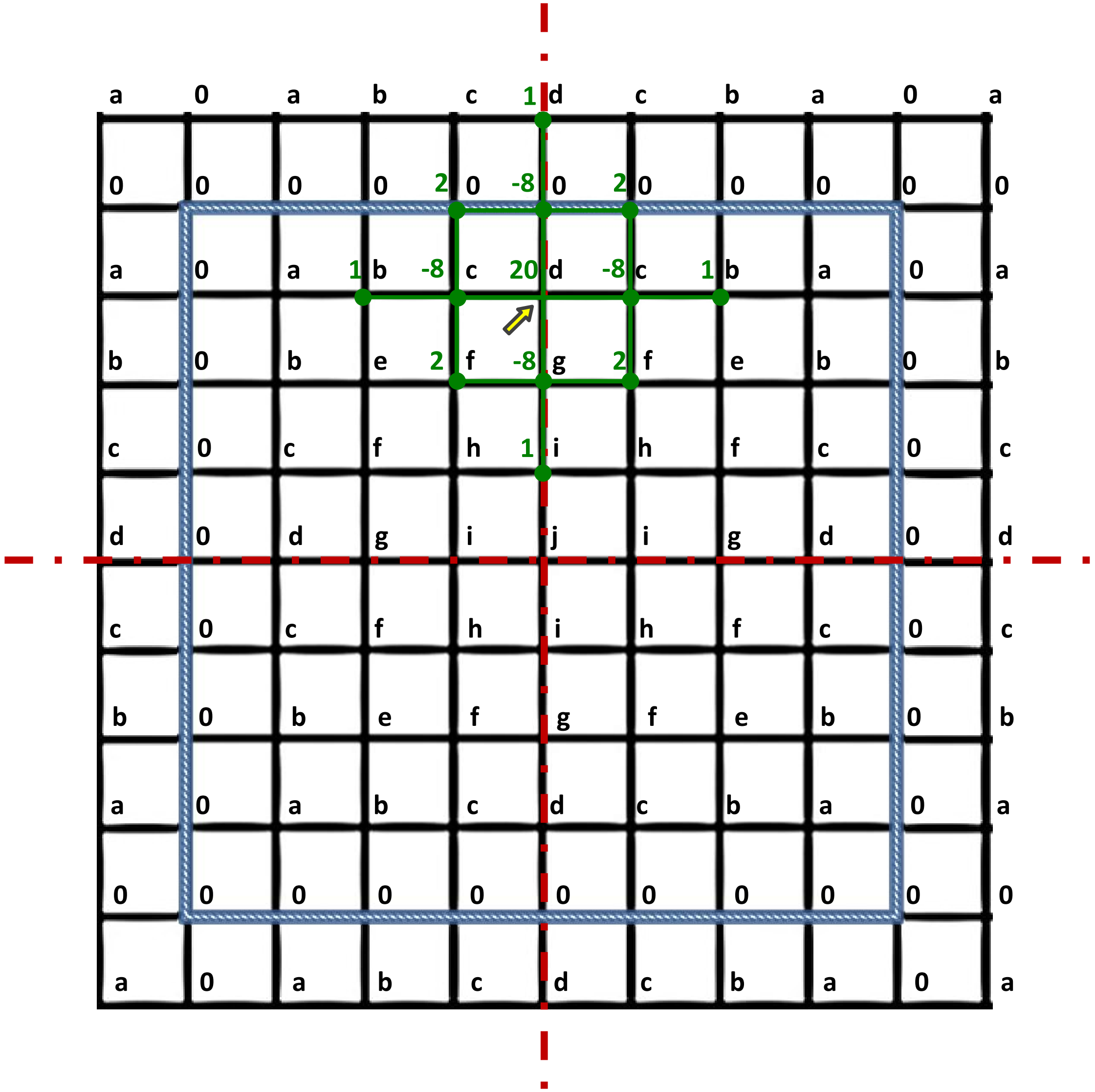


a	0	a	b	c	d	c	b	a	0	a
0	0	0	0	0	0	0	0	0	0	0
a	0	a	b	c	d	c	b	a	0	a
b	0	b	e	f	g	f	e	b	0	b
c	0	c	f	h	i	h	f	c	0	c
d	0	d	g	i	j	i	g	d	0	d
c	0	c	f	h	i	h	f	c	0	c
b	0	b	e	f	g	f	e	b	0	b
a	0	a	b	c	d	c	b	a	0	a
0	0	0	0	0	0	0	0	0	0	0
a	0	a	b	c	d	c	b	a	0	a

▢ clamped edges

$$1\cdot d+2\cdot 0-8\cdot 0+2\cdot 0+1\cdot b-8\cdot c+20\cdot d-8\cdot c+1\cdot b+2\cdot f-8\cdot g+2\cdot f+1\cdot i=\frac{12p(1-\nu^2)L^4}{4096\quad {}^{3}}$$

$$0\cdot a+2\cdot b-16\cdot c+21\cdot d+0\cdot e+4\cdot f-8\cdot g+0\cdot h+1\cdot i+0\cdot j=\frac{12p(1-\nu^2)L^4}{4096E\ {}^{3}}$$

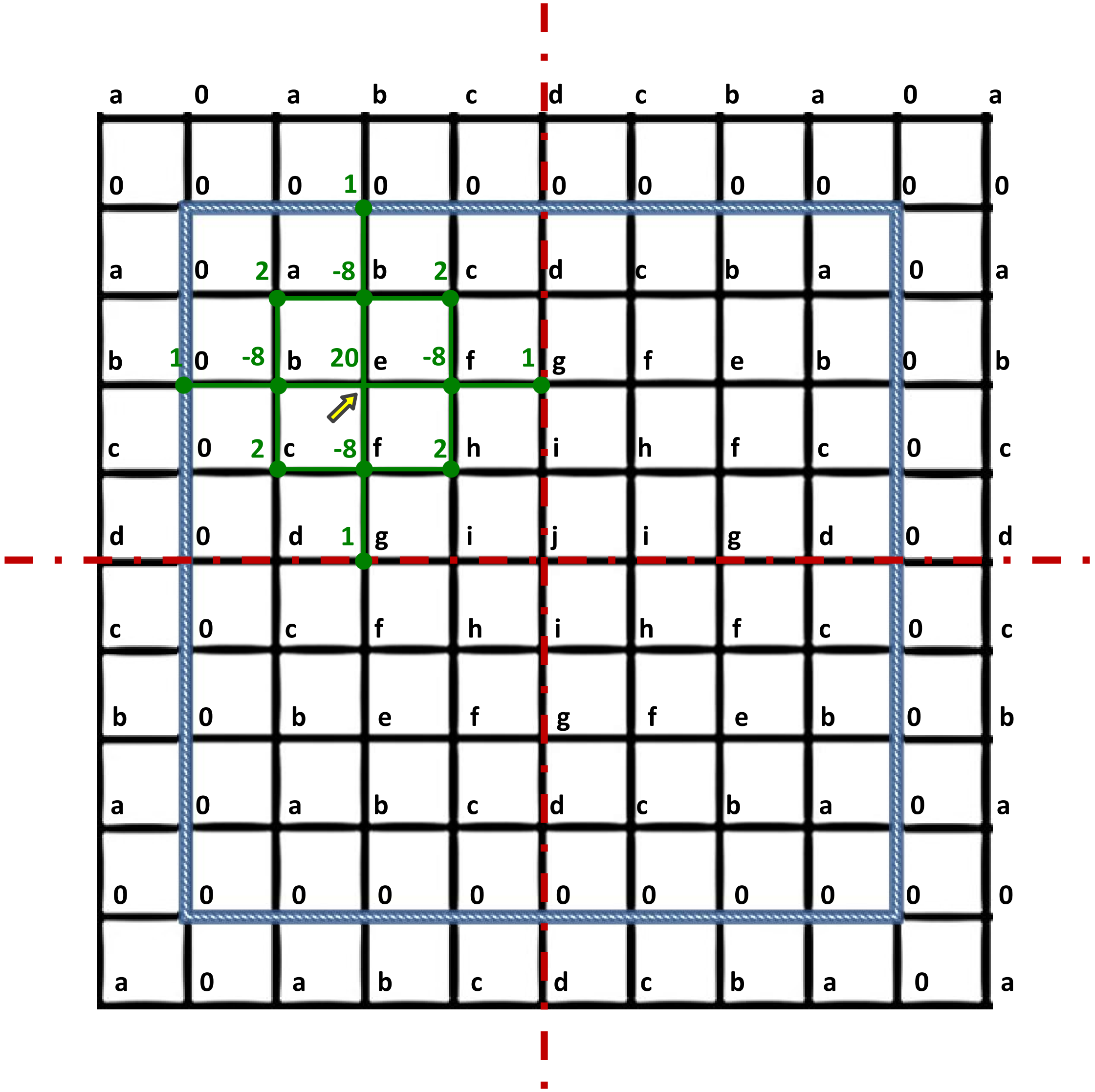


a	0	a	b	c	d	c	b	a	0	a
0	0	0	0	0	0	0	0	0	0	0
a	0	a	b	c	d	c	b	a	0	a
b	0	b	e	f	g	f	e	b	0	b
c	0	c	f	h	i	h	f	c	0	c
d	0	d	g	i	j	i	g	d	0	d
c	0	c	f	h	i	h	f	c	0	c
b	0	b	e	f	g	f	e	b	0	b
a	0	a	b	c	d	c	b	a	0	a
0	0	0	0	0	0	0	0	0	0	0
a	0	a	b	c	d	c	b	a	0	a

▢ clamped edges

$$1\cdot 0+2\cdot a-8\cdot b+2\cdot c+1\cdot 0-8\cdot b+20\cdot e-8\cdot f+1\cdot g+2\cdot c-8\cdot f+2\cdot h+1\cdot g=\frac{12p(1-\nu^2)L^4}{4096Et^3}$$

$$2\cdot a-16\cdot b+4\cdot c+0\cdot d+20\cdot e-16\cdot f+2\cdot g+2\cdot h+0\cdot i+0\cdot j=\frac{12p(1-\nu^2)L^4}{4096Et^3}$$



a	0	a	b	c	d	c	b	a	0	a
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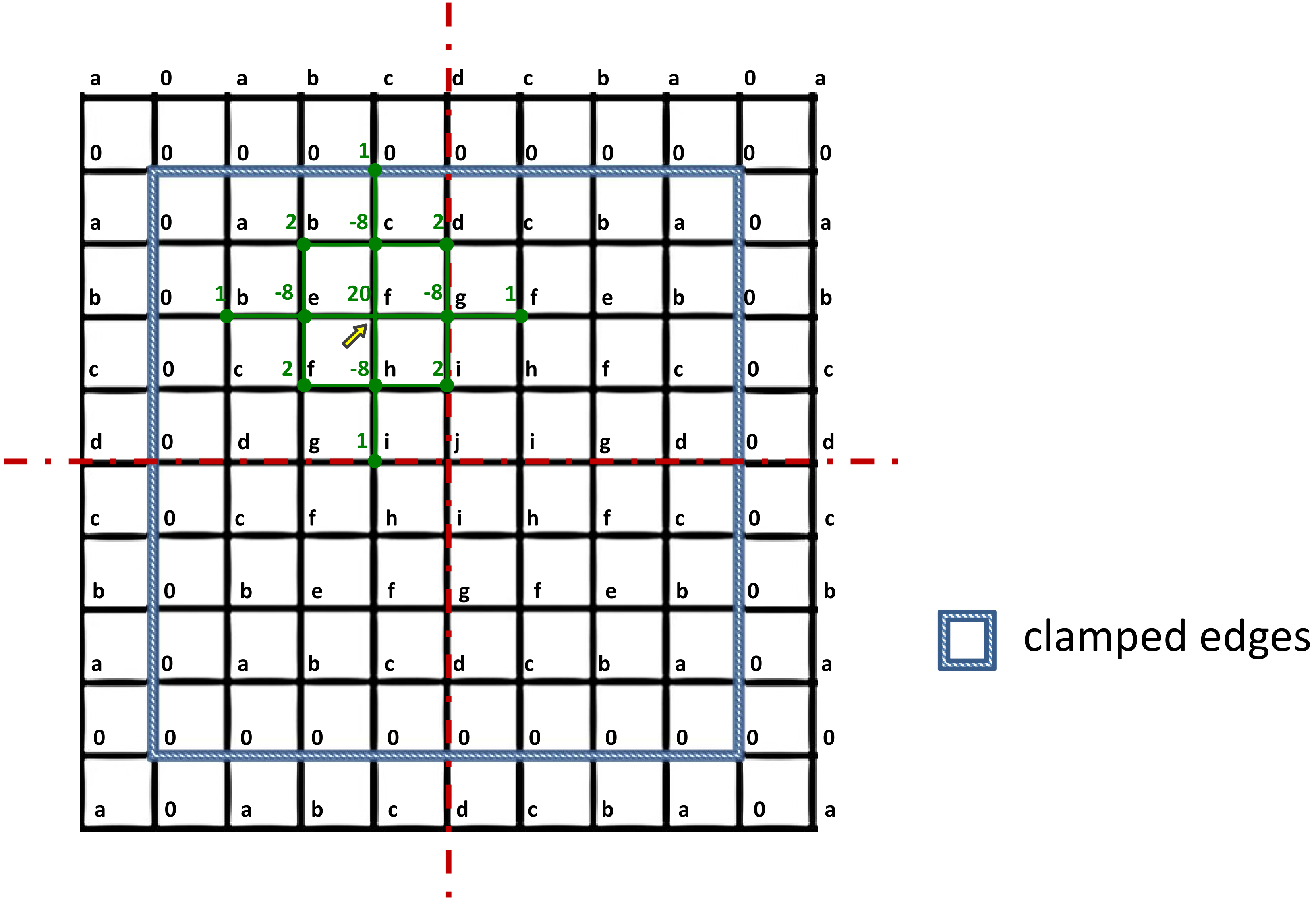
a	0	a	b	c	d	c	b	a	0	a
b	0	b	e	f	g	f	e	b	0	b
c	0	c	f	h	i	h	f	c	0	c
d	0	d	g	i	j	i	g	d	0	d
c	0	c	f	h	i	h	f	c	0	c
b	0	b	e	f	g	f	e	b	0	b
a	0	a	b	c	d	c	b	a	0	a
0	0	0	0	0	0	0	0	0	0	0
a	0	a	b	c	d	c	b	a	0	a

□ clamped edges

$$1 \cdot 0 + 2 \cdot b - 8 \cdot c + 2 \cdot d + 1 \cdot b - 8 \cdot e + 20 \cdot f - 8 \cdot g + 1 \cdot f + 2 \cdot f - 8 \cdot h + 2 \cdot i + 1 \cdot i = \frac{12p(1-\nu^2)L^4}{4096 \quad {}^{3}}$$

$$0 \cdot a + 3 \cdot b - 8 \cdot c + 2 \cdot d - 8 \cdot e + 23 \cdot f - 8 \cdot g - 8 \cdot h + 3 \cdot i + 0 \cdot j = \frac{12p(1-\nu^2)L^4}{4096 \quad {}^{3}}$$



a	0	a	b	c	d	c	b	a	0	a
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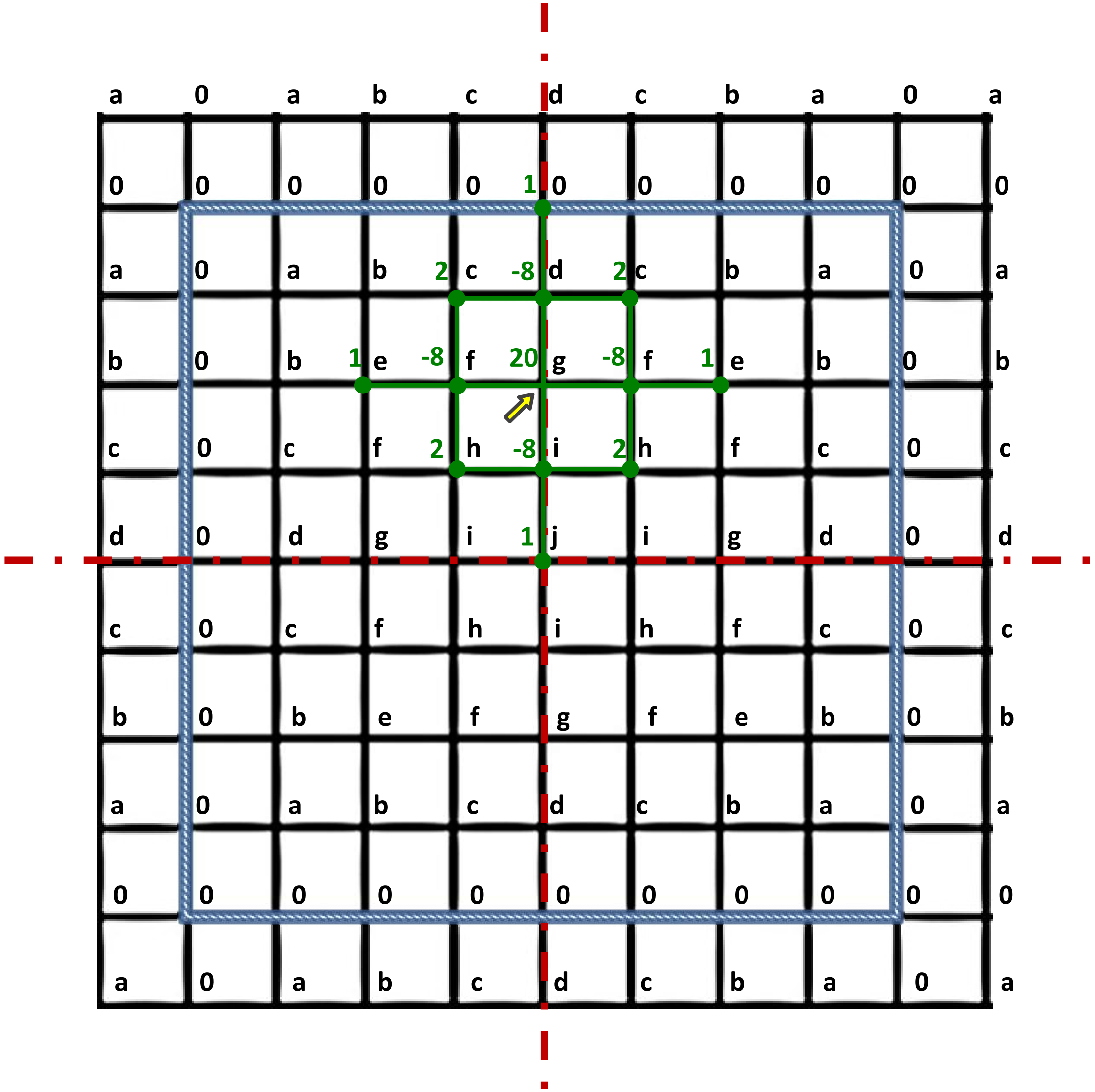
a	0	a	b	c	d	c	b	a	0	a
b	0	b	e	f	g	f	e	b	0	b
c	0	c	f	h	i	h	f	c	0	c
d	0	d	g	i	j	i	g	d	0	d
c	0	c	f	h	i	h	f	c	0	c
b	0	b	e	f	g	f	e	b	0	b
a	0	a	b	c	d	c	b	a	0	a
0	0	0	0	0	0	0	0	0	0	0
a	0	a	b	c	d	c	b	a	0	a

$$1\cdot 0+2\cdot c-8\cdot d+2\cdot c+1\cdot e-8\cdot f+20\cdot g-8\cdot f+1\cdot e+2\cdot h-8\cdot i+2\cdot h+1\cdot j=\frac{12p(1-\nu^2)L^4}{4096Et^3}$$

$$0\cdot a+0\cdot b+4\cdot c-8\cdot d+2\cdot e-16\cdot f+20\cdot g+4\cdot h-8\cdot i+1\cdot j=\frac{12\ \ (1-\nu^2)L^4}{4096Et^3}$$



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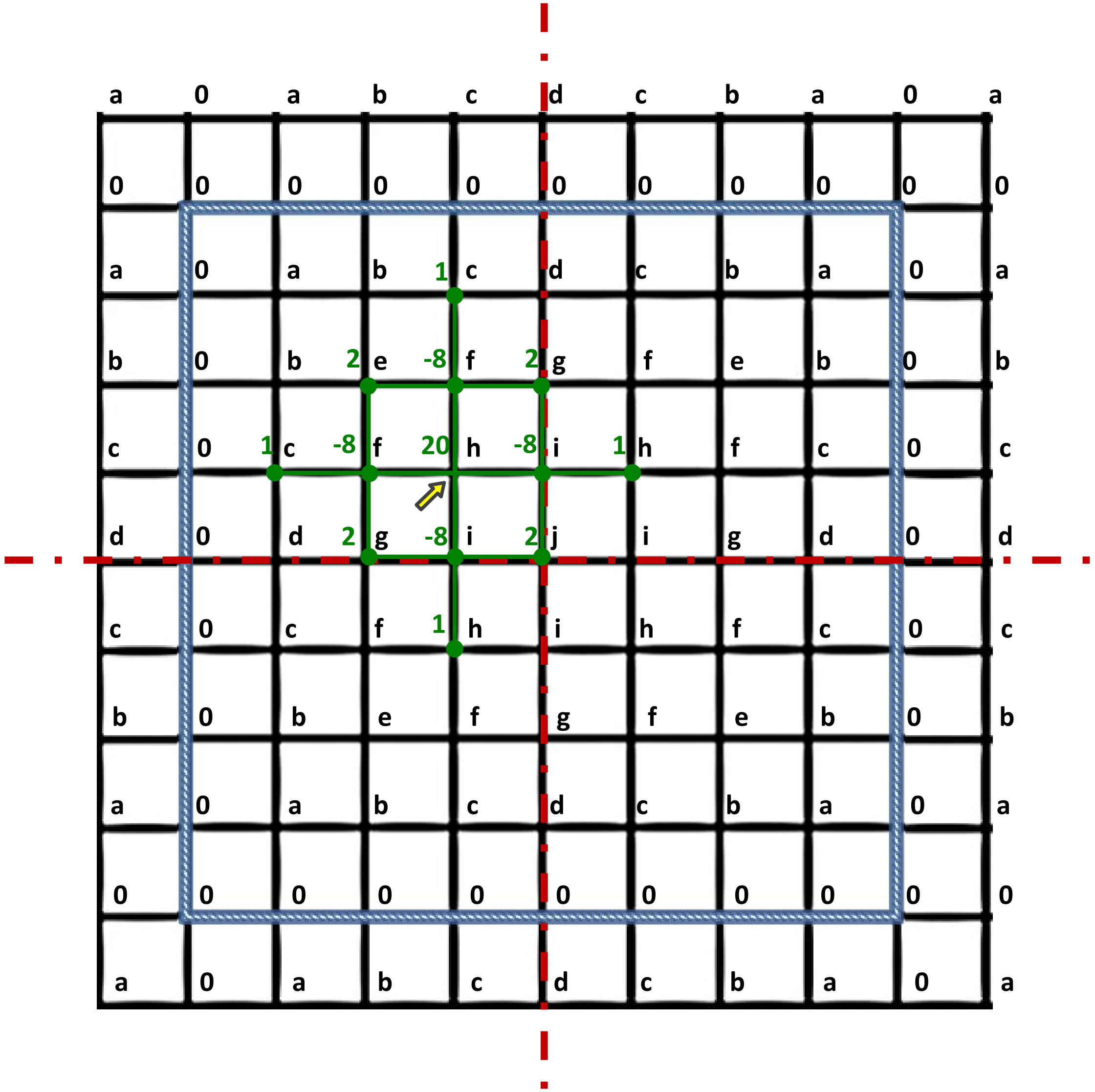
a	0	a	b	c	d	c	b	a	0	a
b	0	b	e	f	g	f	e	b	0	b
c	0	c	f	h	i	h	f	c	0	c
d	0	d	g	i	j	i	g	d	0	d
c	0	c	f	h	i	h	f	c	0	c
b	0	b	e	f	g	f	e	b	0	b
a	0	a	b	c	d	c	b	a	0	a
0	0	0	0	0	0	0	0	0	0	0
a	0	a	b	c	d	c	b	a	0	a

▢ clamped edges

$$1\cdot c+2\cdot e-8\cdot f+2\cdot g+1\cdot c-8\cdot f+20\cdot h-8\cdot i+1\cdot h+2\cdot g-8\cdot i+2\cdot j+1\cdot h=\frac{12\ \ (1-\nu^2)L^4}{4096\ \ \ ^3}$$

$$0\cdot a+0\cdot b+2\cdot c+0\cdot d+2\cdot e-16\cdot f+4\cdot g+22\cdot h-16\cdot i+2\cdot j=\frac{12p(1-\nu^2)L^4}{4096\ \ \ ^3}$$



a	0	a	b	c	d	c	b	a	0	a
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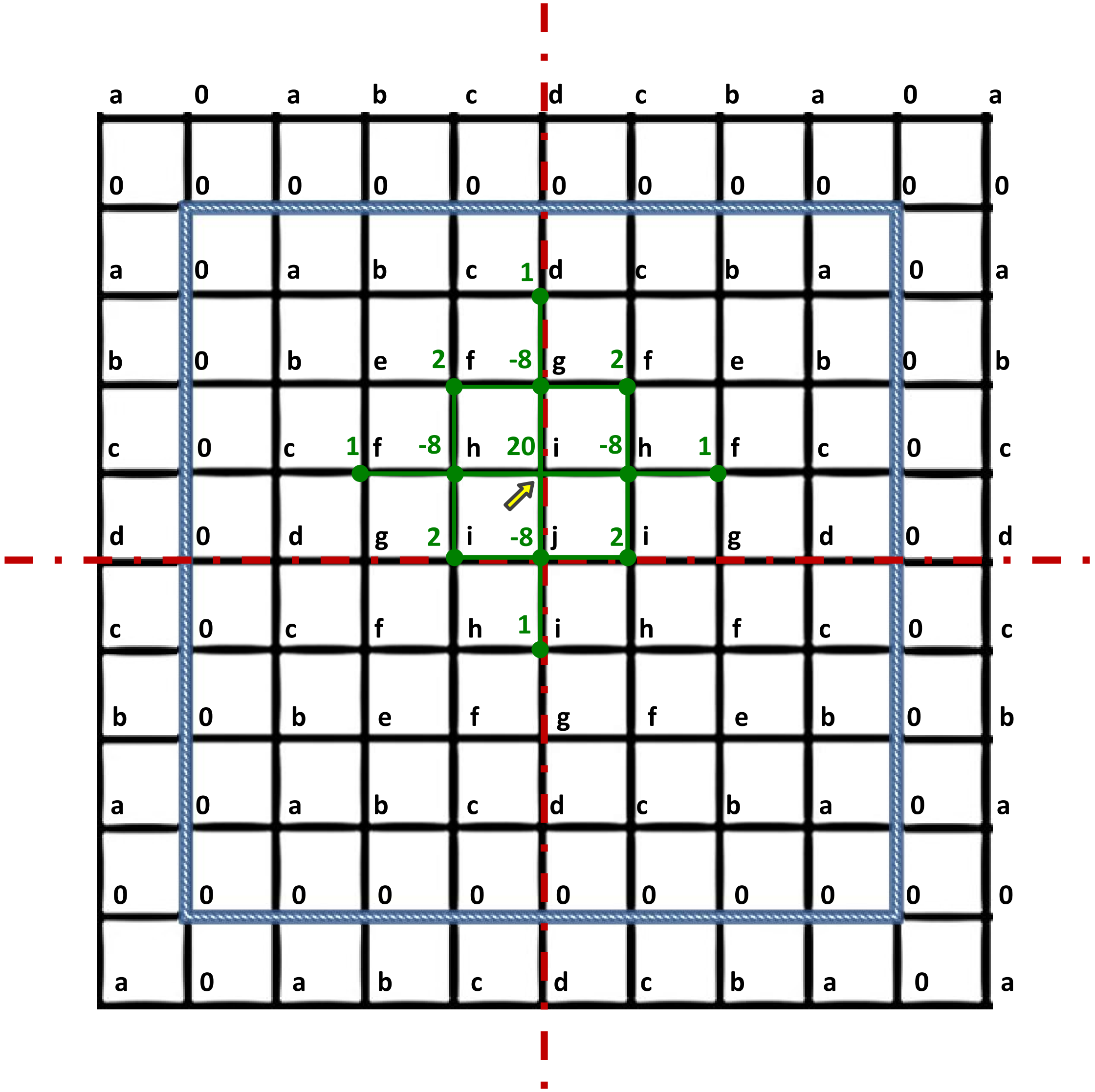
a	0	a	b	c	d	c	b	a	0	a
b	0	b	e	f	g	f	e	b	0	b
c	0	c	f	h	i	h	f	c	0	c
d	0	d	g	i	j	i	g	d	0	d
c	0	c	f	h	i	h	f	c	0	c
b	0	b	e	f	g	f	e	b	0	b
a	0	a	b	c	d	c	b	a	0	a
0	0	0	0	0	0	0	0	0	0	0
a	0	a	b	c	d	c	b	a	0	a

▣ clamped edges

$$1\cdot d + 2\cdot f - 8\cdot g + 2\cdot f + 1\cdot f - 8\cdot h + 20\cdot i - 8\cdot h + 1\cdot f + 2\cdot i - 8\cdot j + 2\cdot i + 1\cdot i = \frac{12p(1-\nu^2)L^4}{4096\quad{}^{3}}$$

$$0\cdot a + 0\cdot b + 0\cdot c + 1\cdot d + 0\cdot e + 6\cdot f - 8\cdot g - 16\cdot h + 25\cdot i - 8\cdot j = \frac{12p(1-\nu^2)L^4}{4096\quad{}^{3}}$$



a	0	a	b	c	d	c	b	a	0	a
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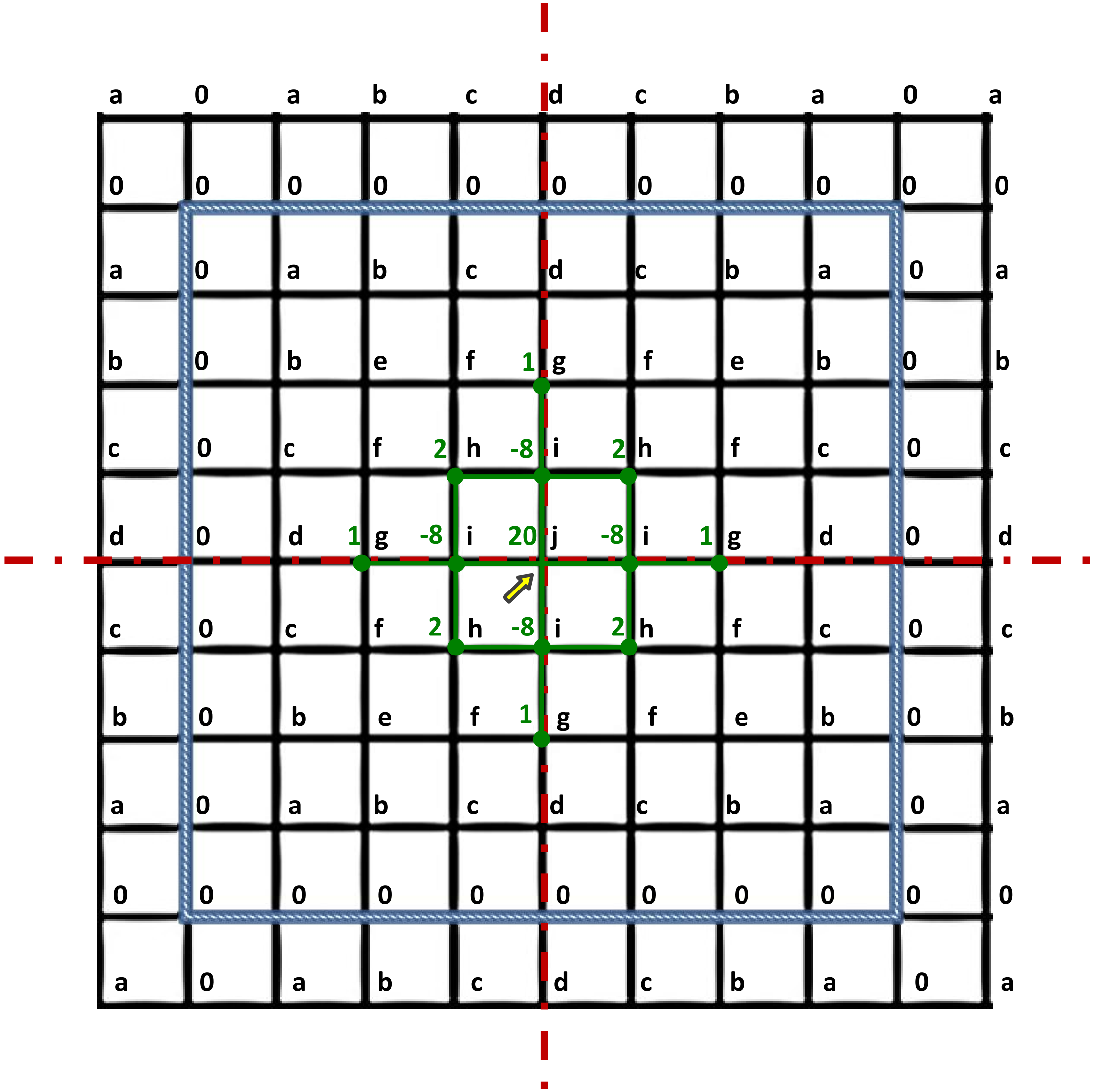
a	0	a	b	c	d	c	b	a	0	a
b	0	b	e	f	g	f	e	b	0	b
c	0	c	f	h	i	h	f	c	0	c
d	0	d	g	i	j	i	g	d	0	d
c	0	c	f	h	i	h	f	c	0	c
b	0	b	e	f	g	f	e	b	0	b
a	0	a	b	c	d	c	b	a	0	a
0	0	0	0	0	0	0	0	0	0	0
a	0	a	b	c	d	c	b	a	0	a

▣ clamped edges

$$1\cdot g+2\cdot h-8\cdot i+2\cdot h+1\cdot g-8\cdot i+20\cdot j-8\cdot i+1\cdot g+2\cdot h-8\cdot i+2\cdot h+1\cdot g=\frac{12p(1-\nu^2)L^4}{4096Et^3}$$

$$0\cdot a+0\cdot b+0\cdot c+0\cdot d+0\cdot e+0\cdot f+4\cdot g+8\cdot h-32\cdot i+20\cdot j=\frac{12\ \ (1-\nu^2)L^4}{4096Et^3}$$



a	0	a	b	c	d	c	b	a	0	a
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a	0	a	b	c	d	c	b	a	0	a
b	0	b	e	f	g	f	e	b	0	b
c	0	c	f	h	i	h	f	c	0	c
d	0	d	g	i	j	i	g	d	0	d
c	0	c	f	h	i	h	f	c	0	c
b	0	b	e	f	g	f	e	b	0	b
a	0	a	b	c	d	c	b	a	0	a
0	0	0	0	0	0	0	0	0	0	0
a	0	a	b	c	d	c	b	a	0	a

▣ clamped edges

$$22 \cdot a - 16 \cdot b + 2 \cdot c + 0 \cdot d + 2 \cdot e + 0 \cdot f + 0 \cdot g + 0 \cdot h + 0 \cdot i + 0 \cdot j = \frac{12\ \ (1-\nu^2)L^4}{4096Et^3}$$

$$-8 \cdot a + 23 \cdot b - 8 \cdot c + 1 \cdot d - 8 \cdot e + 3 \cdot f + 0 \cdot g + 0 \cdot h + 0 \cdot i + 0 \cdot j = \frac{12p(1-\nu^2)L^4}{4096Et^3}$$

$$1 \cdot a - 8 \cdot b + 22 \cdot c - 8 \cdot d + 2 \cdot e - 8 \cdot f + 2 \cdot g + 1 \cdot h + 0 \cdot i + 0 \cdot j = \frac{12p(1-\nu^2)L^4}{4096Et^3}$$

$$0 \cdot a + 2 \cdot b - 16 \cdot c + 21 \cdot d + 0 \cdot e + 4 \cdot f - 8 \cdot g + 0 \cdot h + 1 \cdot i + 0 \cdot j = \frac{12p(1-\nu^2)L^4}{4096\ \ \ ^3}$$

$$2 \cdot a - 16 \cdot b + 4 \cdot c + 0 \cdot d + 20 \cdot e - 16 \cdot f + 2 \cdot g + 2 \cdot h + 0 \cdot i + 0 \cdot j = \frac{12p(1-\nu^2)L^4}{4096E\ ^3}$$

$$0 \cdot a + 3 \cdot b - 8 \cdot c + 2 \cdot d - 8 \cdot e + 23 \cdot f - 8 \cdot g - 8 \cdot h + 3 \cdot i + 0 \cdot j = \frac{12p(1-\nu^2)L^4}{4096Et^3}$$

$$0 \cdot a + 0 \cdot b + 4 \cdot c - 8 \cdot d + 2 \cdot e - 16 \cdot f + 20 \cdot g + 4 \cdot h - 8 \cdot i + 1 \cdot j = \frac{12\ \ (1-\nu^2)L^4}{4096\ \ \ ^3}$$

$$0 \cdot a + 0 \cdot b + 2 \cdot c + 0 \cdot d + 2 \cdot e - 16 \cdot f + 4 \cdot g + 22 \cdot h - 16 \cdot i + 2 \cdot j = \frac{12p(1-\nu^2)L^4}{4096\ \ \ ^3}$$

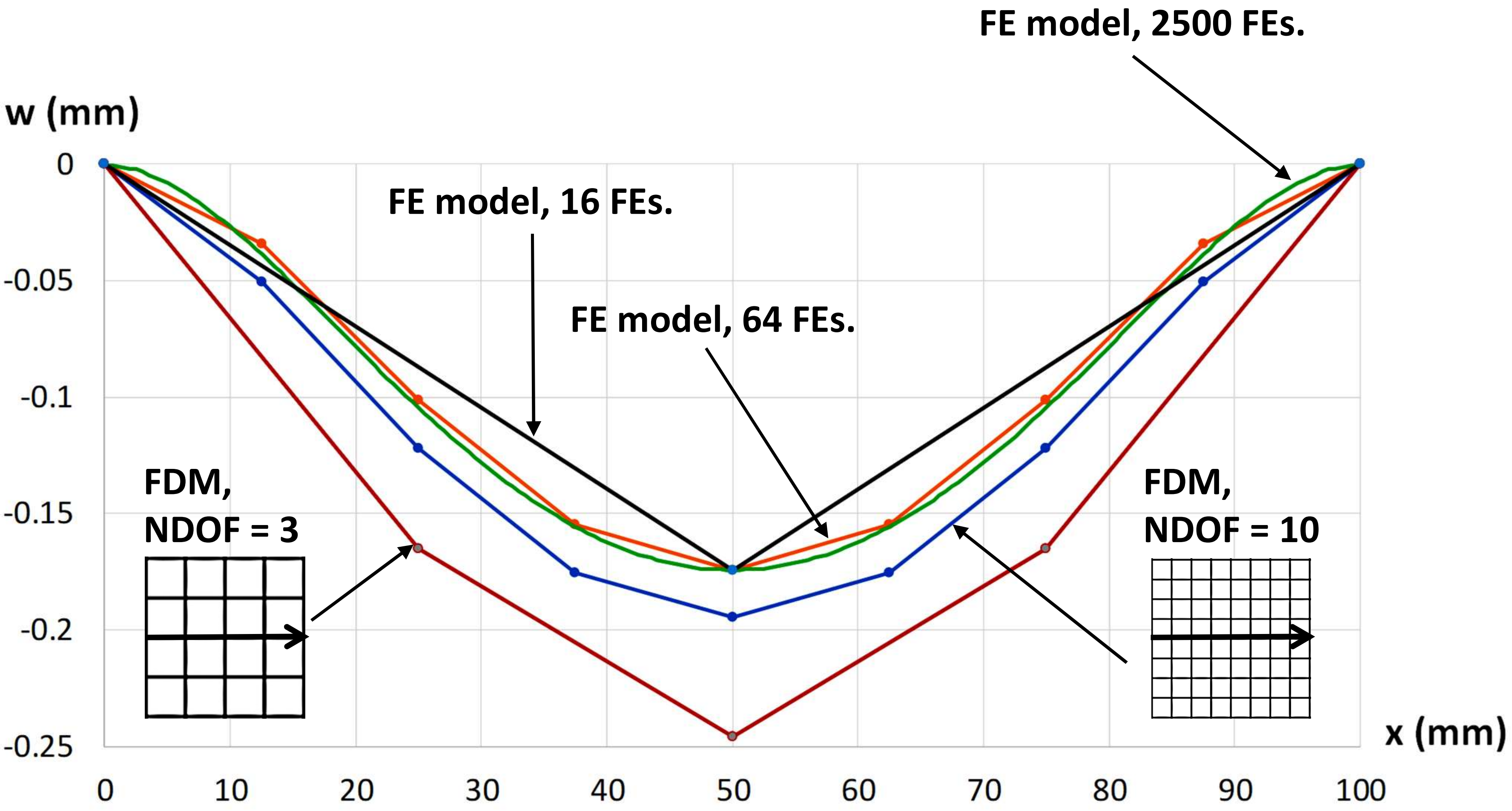
$$0 \cdot a + 0 \cdot b + 0 \cdot c + 1 \cdot d + 0 \cdot e + 6 \cdot f - 8 \cdot g - 16 \cdot h + 25 \cdot i - 8 \cdot j = \frac{12p(1-\nu^2)L^4}{4096Et^3}$$

$$0 \cdot a + 0 \cdot b + 0 \cdot c + 0 \cdot d + 0 \cdot e + 0 \cdot f + 4 \cdot g + 8 \cdot h - 32 \cdot i + 20 \cdot j = \frac{12p(1-\nu^2)L^4}{4096Et^3}$$

$$\begin{bmatrix} 22 & -16 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ -8 & 23 & -8 & 1 & -8 & 3 & 0 & 0 & 0 & 0 \\ 1 & -8 & 22 & -8 & 2 & -8 & 2 & 1 & 0 & 0 \\ 0 & 2 & -16 & 21 & 0 & 4 & -8 & 0 & 1 & 0 \\ 2 & -16 & 4 & 0 & 20 & -16 & 2 & 2 & 0 & 0 \\ 0 & 3 & -8 & 2 & -8 & 23 & -8 & -8 & 3 & 0 \\ 0 & 0 & 4 & -8 & 2 & -16 & 20 & 4 & -8 & 1 \\ 0 & 0 & 2 & 0 & 2 & -16 & 4 & 22 & -16 & 2 \\ 0 & 0 & 0 & 1 & 0 & 6 & -8 & -16 & 25 & -8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 8 & -32 & 20 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ i \\ j \end{Bmatrix} = \frac{12p(1-\nu^2)L^4}{4096Et^3} \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$$

Point	a	b	c	d	e	f	g	h	i	j
w (mm)	-0.014	-0.033	-0.046	-0.051	-0.077	-0.110	-0.122	-0.158	-0.175	-0.194

Deformation of a square plate: Finite Difference Method and FEM solution (4 node SHELL 181)



Bending moments

$$m_x = -\frac{Et^3}{12(1-\nu^2)}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

$$m_y = -\frac{Et^3}{12(1-\nu^2)}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)$$

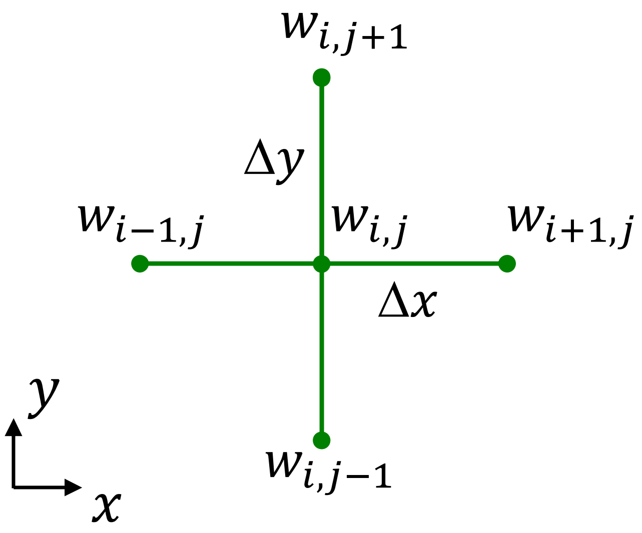
Normal stress components on the top layer (*z* = ½*t*)

$$\sigma_x = \frac{6m_x}{t^2} \qquad ; \qquad \sigma_y = \frac{6m_y}{t^2}$$

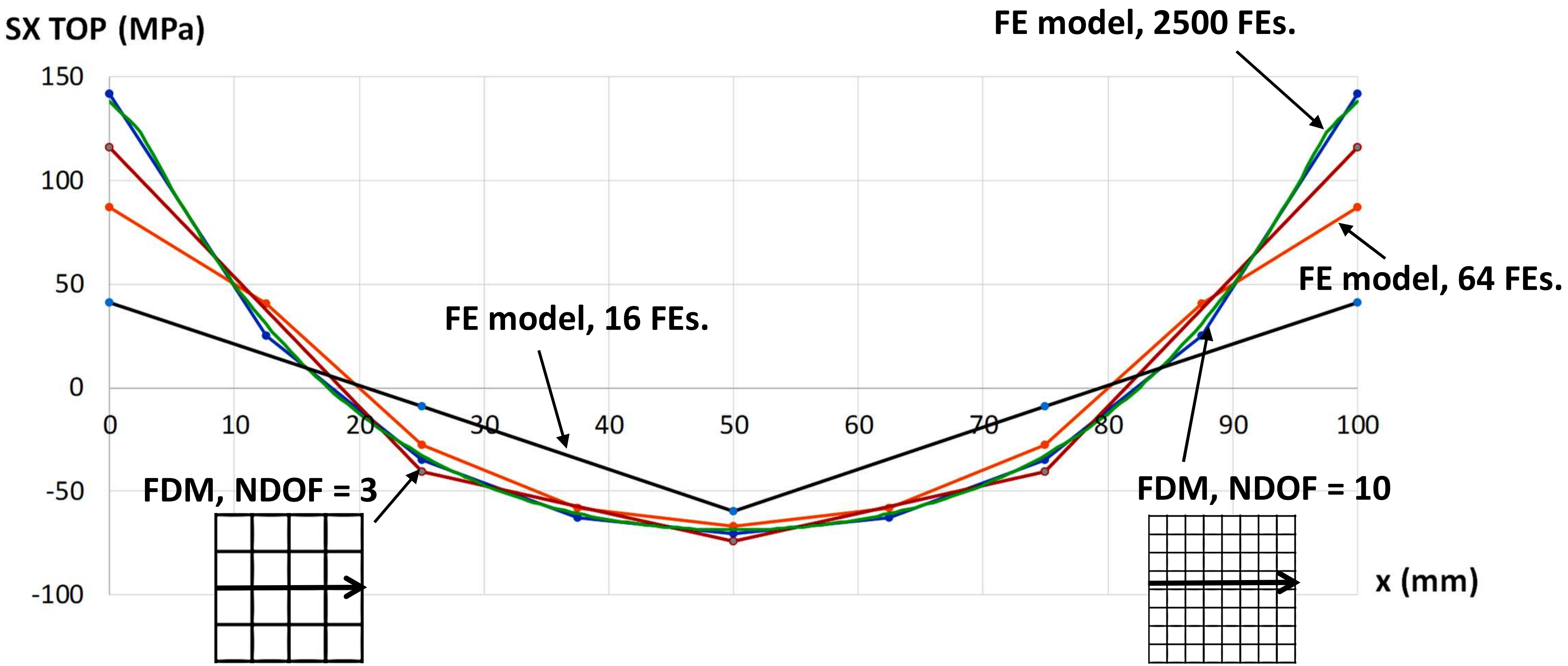
Curvatures:

$$\frac{\partial^2 w}{\partial x^2} = \frac{w_{i+1,j} - 2w_{i,j} + w_{i-1,j}}{\Delta x^2}$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{w_{i,j+1} - 2w_{i,j} + w_{i,j-1}}{\Delta y^2}$$



Stress SX component on the top layer: Finite Difference Method and FEM solution (4 node SHELL 181)



Stress SY component on the top layer: Finite Difference Method and FEM solution (4 node SHELL 181)