Project no. 5 (12)
Phugoid (Long Period) Motion of the Airplane

12.1 Introduction

The *phugoid* or *long period motion* is a characteristic oscillations of the aircraft after a small disturbance of the steady flight (ie. due to small horizontal control surface motion or the air gust). The airplane is traveling along the sinusoidal trajectory with small changes of the air speed and pitch angle.

12.2 Equations of motion

Let assume:

- an aircraft is initially flying in vertical plane with constant speed V_0 and with no rotation,
- after small disturbance of the flight speed $V=V_0 + v$ or pitch angle θ , the airplane will be always in the vertical plane,
- the airplane have two degree of freedom,
- the wing angle of attack as well as aerodynamic coefficients C_L and C_D are constant.

It can be shown that equations of motion of the airplane developed using Frenet coordinates system are:

$$\begin{aligned} m\cdot\frac{dv}{dt}+m\cdot g\cdot\sin\theta+\frac{1}{2}\cdot\rho\cdot S_w\cdot(V_0+v)^2\cdot C_D=0 \ , \\ m\cdot V_0\frac{d\theta}{dt}+m\cdot g\cdot\cos\theta+\frac{1}{2}\cdot\rho\cdot S_w\cdot(V_0+v)^2\cdot C_L=0 \ . \end{aligned} \tag{1}$$

Dimensionless and linear form of equations derived from (1) is:

$$\begin{aligned} \frac{d\bar{v}}{d\bar{t}}+\frac{1}{2}\cdot C_L\cdot\theta+C_D\cdot\bar{v} = 0 \ , \\ \frac{d\bar{\theta}}{d\bar{t}}+\frac{1}{2}\cdot C_D\cdot\theta-C_L\cdot\bar{v} = 0 \ , \end{aligned} \tag{2}$$

where:

$\bar{v}=\dfrac{v}{V_0}$ - non-dimensional small disturbance of flight speed,

$\bar{\theta}$ - small disturbance of the pitch angle (in radians),

$\bar{t}=\dfrac{t}{t_{aero}}$ - dimensionless time,

$t_{aero}=\dfrac{2\cdot m}{\rho\cdot S_w\cdot V_0}$ - aerodynamic time.

Expected solutions of non-dimensional equations (2) can be written as

$$\bar{v}(\bar{t})=\bar{v}_0\cdot e^{\bar{\lambda}\cdot\bar{t}} \ , \qquad \bar{\theta}(\bar{t})=\bar{\theta}_0\cdot e^{\bar{\lambda}\cdot\bar{t}} \tag{3}$$

Substituting above function to equations of motion we obtain following set of linear equations (4) with unknown constant parameters $\bar{v}_0, \bar{\theta}_0, \bar{\lambda}$ called as *eigenvector* $[\bar{v}_0, \bar{\theta}_0]^T$ and *eigenvalue* $\bar{\lambda}$ of the dynamic system:

$$\begin{bmatrix} \bar{\lambda}+C_D & \frac{1}{2}\cdot C_L \\ -C_L & \bar{\lambda}+\frac{1}{2}\cdot C_D \end{bmatrix}\cdot\begin{bmatrix} \bar{v}_0 \\ \bar{\theta}_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4}$$

The equations (4) can be solved with non-zero solution of the eigenvector if the determinant of the first matrix in (4) is equal to zero:

$$det\left(\begin{bmatrix} \bar{\lambda}+C_D & \frac{1}{2}\cdot C_L \\ -C_L & \bar{\lambda}+\frac{1}{2}\cdot C_D \end{bmatrix}\right) = 0 \tag{5}$$

and this condition give us the quadratic algebraic equation for eigenvalues λ :

$$\bar{\lambda}^2+\frac{3}{2}\cdot C_D\cdot\bar{\lambda}+\frac{1}{2}\cdot\left(C_D^2+C_L^2\right) = 0. \tag{6}$$

with two solutions for eigenvalues as a pair of complex coupled numbers as follows:

$$\bar{\lambda}_{1,2} = \bar{\xi}_{1,2} \pm i\ \bar{\eta}_{1,2} = -\frac{3}{4}\cdot C_D \pm i\sqrt{\frac{1}{2}\cdot\left(C_D^2+C_L^2\right)-\left(\frac{3}{4}\cdot C_D\right)^2}. \tag{7}$$

Complete solution of the non-dimensional dynamic equations of motion (2) is a linear combination of two sets of expected solutions (3):

$$\begin{aligned} \bar{v}(\bar{t}) &= \bar{v}_1\cdot e^{\bar{\lambda}_1\cdot\bar{t}}+\bar{v}_2\cdot e^{\bar{\lambda}_2\cdot\bar{t}} = e^{\bar{\xi}_{1,2}}\cdot\left[\left(\bar{v}_1+\bar{v}_2\right)\cdot\cos\left(\bar{\eta}_{1,2}\cdot\bar{t}\right) + i\cdot\left(\bar{v}_1-\bar{v}_2\right)\cdot\sin\left(\bar{\eta}_{1,2}\cdot\bar{t}\right)\right], \\ \bar{\theta}(\bar{t}) &= \bar{\theta}_1\cdot e^{\bar{\lambda}_1\cdot\bar{t}}+\bar{\theta}_2\cdot e^{\bar{\lambda}_2\cdot\bar{t}} = e^{\bar{\xi}_{1,2}}\cdot\left[\left(\bar{\theta}_1+\bar{\theta}_2\right)\cdot\cos\left(\bar{\eta}_{1,2}\cdot\bar{t}\right) + i\cdot\left(\bar{\theta}_1-\bar{\theta}_2\right)\cdot\sin\left(\bar{\eta}_{1,2}\cdot\bar{t}\right)\right]. \end{aligned} \tag{8}$$

where:

$\bar{v}_1$, $\bar{v}_2$, $\bar{\theta}_1$, $\bar{\theta}_2$ - non-dimensional amplitudes of oscillations (elements of two eigenvectors "1" and "2"),

$\bar{\xi}_{1,2}=\mathrm{Re}\left(\bar{\lambda}_{1,2}\right) = -\frac{3}{4}\cdot C_D$ - non-dimensional damping coefficient,

$\bar{\eta}_{1,2}=\mathrm{Im}\left(\bar{\lambda}_{1,2}\right) = \sqrt{\frac{1}{2}\cdot\left(C_D^2+C_L^2\right)-\left(\frac{3}{4}\cdot C_D\right)^2}$ - non-dimensional frequency of oscillations.

Elements of eigenvectors can be calculated by substituting into (4) subsequently eigenvalues (7) λ_1 and λ_2 :

$$\left(\frac{\bar{v}}{\bar{\theta}}\right)_{1,2} = -\frac{2\cdot\left(\bar{\lambda}_{1,2}+C_D\right)}{C_L} = a_{1,2}\pm i\cdot b_{1,2}, \tag{9}$$

and the solution (8) can be written as:

$$\begin{aligned} \bar{v}(\bar{t}) &= e^{\bar{\xi}_{1,2}}\cdot\left[\left(\bar{v}_1+\bar{v}_2\right)\cdot\cos\left(\bar{\eta}_{1,2}\cdot\bar{t}\right) + i\cdot\left(\bar{v}_1-\bar{v}_2\right)\cdot\sin\left(\bar{\eta}_{1,2}\cdot\bar{t}\right)\right], \\ \bar{\theta}(\bar{t}) &= e^{\bar{\xi}_{1,2}}\cdot\left[\left(\frac{\bar{\theta}_1}{\bar{v}_1}\right)\cdot\bar{v}_1\cdot\cos\left(\bar{\eta}_{1,2}\cdot\bar{t}\right) - i\cdot\left(\frac{\bar{\theta}_2}{\bar{v}_2}\right)\cdot\bar{v}_2\cdot\sin\left(\bar{\eta}_{1,2}\cdot\bar{t}\right)\right]. \end{aligned} \tag{10}$$

Note that in the solution (10) there are now two unknown constant - $\bar{v}_1$, $\bar{v}_2$. It can be calculated using initial conditions for (2):

$$\bar{t}=\bar{t}_0=0: \quad \bar{v}(0)=\bar{u}_{0,} \quad \bar{\theta}(0)=\bar{\theta}_0.$$

For example, if $\bar{\theta}_0 = 0$ (we assume small disturbance of the flight speed only) then the final solution for this case is:

$$\bar{v}(\bar{t}) = \frac{\bar{u}_0}{b_{1,2}} \cdot \sqrt{a_{1,2}^2 + b_{1,2}^2} \cdot e^{\bar{\xi}_{1,2}} \cdot \cos\left(\bar{\eta}_{1,2} \cdot \bar{t} + \bar{\varphi}_0\right) ,$$
$$\bar{\theta}(\bar{t}) = \frac{\bar{u}_0}{b_{1,2}} \cdot \left(a_{1,2}^2 + b_{1,2}^2\right) \cdot e^{\bar{\xi}_{1,2}} \cdot \sin\left(\bar{\eta}_{1,2} \cdot \bar{t}\right) , \tag{11}$$

where (see (9)):

$$a_{1,2} = -2 \cdot \frac{\bar{\xi}_{1,2} + C_D}{C_L} = -\frac{C_D}{2 \cdot C_L} < 0, \qquad b_{1,2} = \frac{2 \cdot \bar{\eta}_{1,2}}{C_L} > 0 , \qquad \bar{\varphi}_0 = atan \frac{a_{1,2}}{b_{1,2}} . \tag{12}$$

Figures 1 and 2 show results of calculations using solutions (7) and (11) for a small aircraft (in-flight mass = 1000 kg, wing area S = 10 m^2 , parabolic polar $C_D = 0.03 + 0.025 * C_L^2$): phugoid eigenvalues (non-dimensional) as well as period and time of half damping of the oscillation amplitude as functions of flight speed.

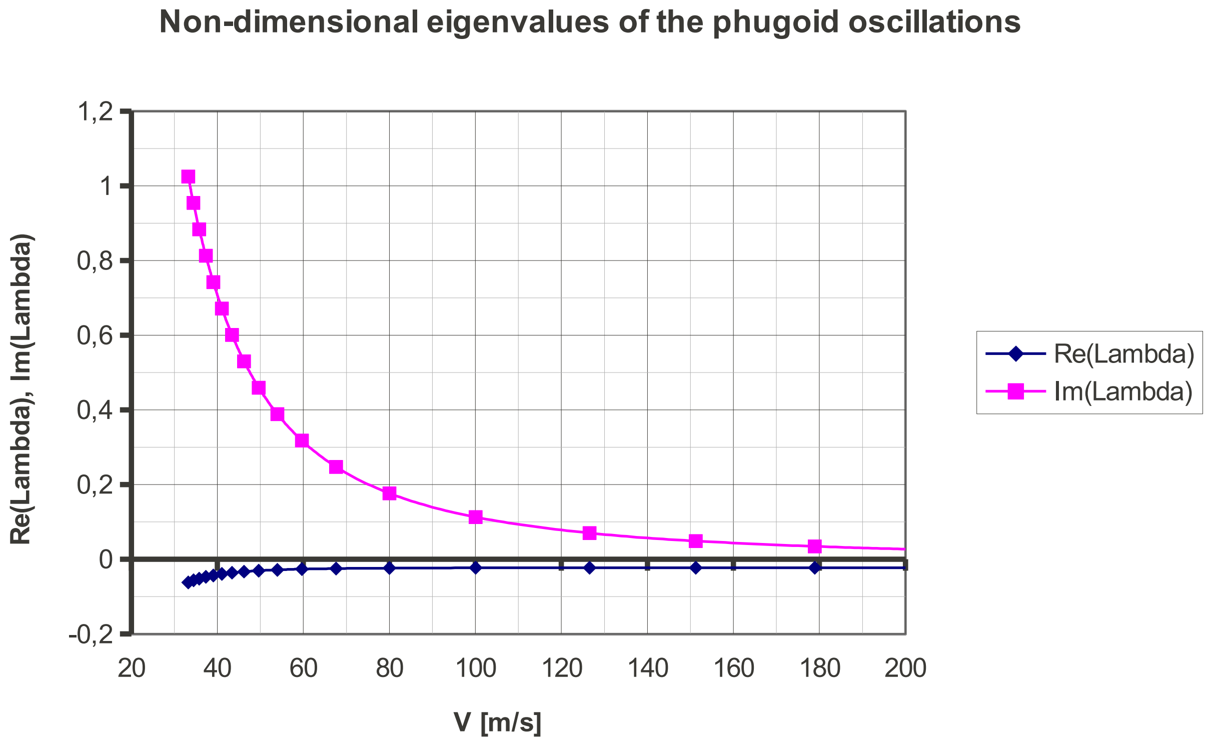

Figure 1

Figure 3 presents the (dimensional!) response of the aircraft for small perturbation of the flight speed by $u_0 = 0.5$ m/s (equations (11)). Initial flight speed $V_0 = 50$ m/s, initial pitch angle $\theta_0 = 0$ at h = 0 m (close to the ground).

Note the long period of the oscillations (45 seconds), small damping of the oscillations (time to half amplitude 75 seconds) as well as small values of changes the speed V and pitch angle Θ .

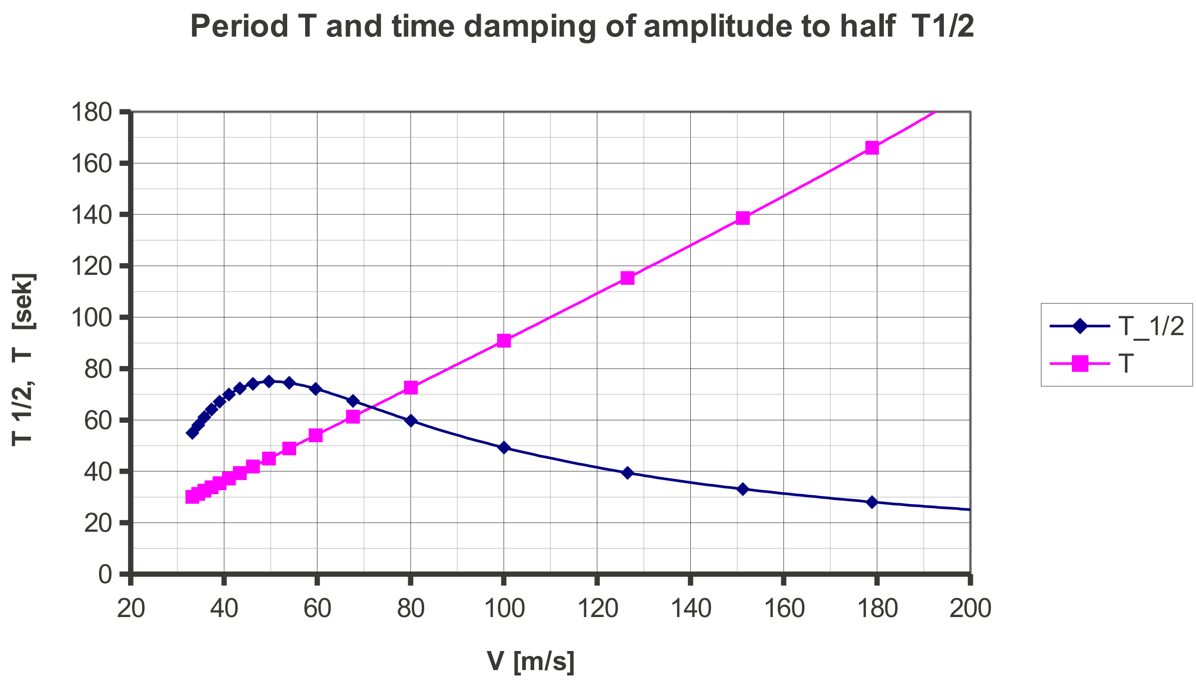

Figure 2

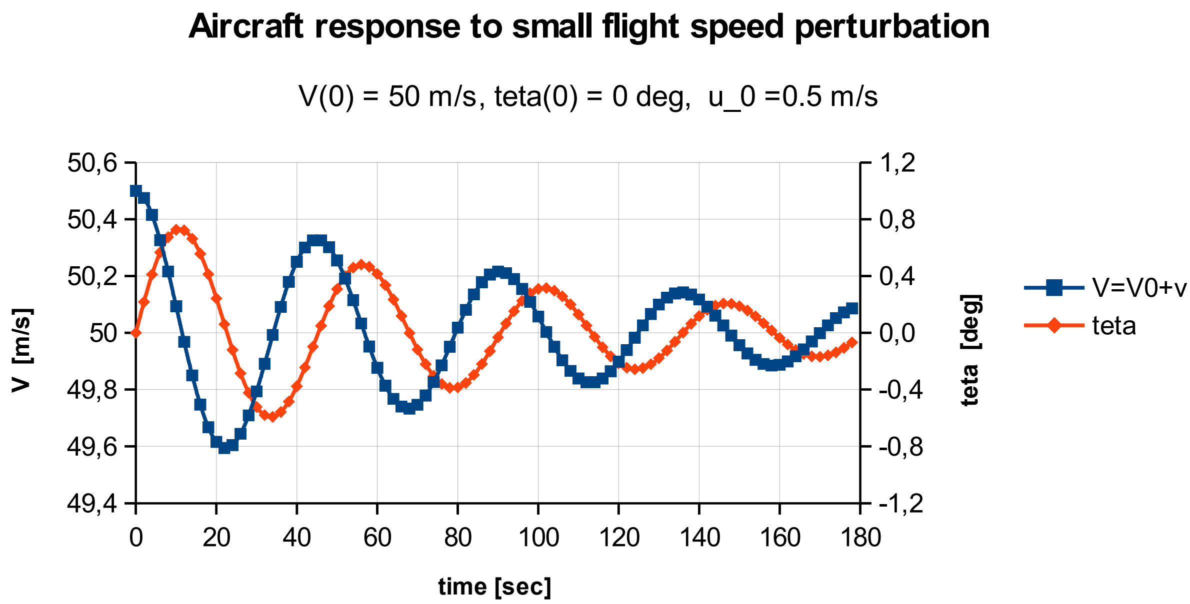

Figure 3