

# LECTURE 11

# ENERGY OF A FLUID FLOW



## DERIVATION OF THE ENERGY EQUATION

According to the general conservation law formula introduced in the **Lecture 3**, the formula of the conservation of the total energy of fluid contained in the control volume  $\Omega$  can be written in the following form

$$\left. \frac{d}{dt} \int_{\Omega} \rho e dV \right|_{\text{production}} = \underbrace{P_S + P_V}_{\text{power of external forces}} + \underbrace{Q_{\partial\Omega}}_{\text{conduction of heat through } \partial\Omega} + \underbrace{Q_{\Omega}}_{\text{internal heat sources}}$$

where  $e = u + \frac{1}{2}v^2$  denotes the mass-specific density of the total energy.

The terms describing the power of surface and volumetric forces are

$$P_S = \int_{\partial\Omega} \boldsymbol{\sigma} \cdot \boldsymbol{v} dS = \int_{\partial\Omega} \boldsymbol{\Xi} \boldsymbol{n} \cdot \boldsymbol{v} dS \quad , \quad P_V = \int_{\Omega} \rho \boldsymbol{f} \cdot \boldsymbol{v} dV$$

while the heat terms are defined as

$$Q_{\partial\Omega} = - \int_{\partial\Omega} \boldsymbol{q}_h \cdot \boldsymbol{n} dS \quad , \quad Q_{\Omega} = \int_{\Omega} \rho \gamma_h dV$$

In the above, the symbol  $\boldsymbol{q}_h$  denotes the vector of conductive heat flux through the boundary  $\partial\Omega$  and the symbol  $\gamma_h$  stands for the mass-specific density of internal heat sources.

The left-hand side of the above integral formula can be further transform as follows

$$\begin{aligned}
 \frac{d}{dt} E \Big|_{production} &\equiv \int_{\Omega} \frac{\partial}{\partial t} [\rho(u + \frac{1}{2}v^2)] dV + \int_{\partial\Omega} [\rho(u + \frac{1}{2}v^2)] \mathbf{v} \cdot \mathbf{n} dS = \\
 &= \int_{\Omega} \frac{\partial}{\partial t} [\rho(u + \frac{1}{2}v^2)] dV + \int_{\Omega} \nabla \cdot [\rho(u + \frac{1}{2}v^2) \mathbf{v}] dV = \\
 &= \int_{\Omega} \left\{ (u + \frac{1}{2}v^2) \left[ \frac{\partial}{\partial t} \rho + \nabla \cdot (\rho \mathbf{v}) \right] + \rho \left[ \frac{\partial}{\partial t} (u + \frac{1}{2}v^2) + \mathbf{v} \cdot \nabla (u + \frac{1}{2}v^2) \right] \right\} dV = \\
 &= \int_{\Omega} \rho \left[ \frac{\partial}{\partial t} (u + \frac{1}{2}v^2) + \mathbf{v} \cdot \nabla (u + \frac{1}{2}v^2) \right] dV = \int_{\Omega} \rho \frac{D}{Dt} (u + \frac{1}{2}v^2) dV = \int_{\Omega} \rho \frac{D}{Dt} e dV
 \end{aligned}$$

where the expression marked in green is zero due to the mass conservation. The substantial derivative of the mass-specific energy  $e$  has appeared.

Our next step is (as usual) to transform the source terms expressed by the surface integrals into the form which contains volume integrals. Consider the terms describing the power performed by surface stress. This what we do ...

$$\begin{aligned}
P_S &= \int_{\partial\Omega} \mathbf{E}\mathbf{n} \cdot \mathbf{v} \, dS = \int_{\partial\Omega} v_i \mathbf{E}_{ij} n_j \, dS \stackrel{GGO}{=} \int_{\Omega} \frac{\partial}{\partial x_j} (v_i \mathbf{E}_{ij}) \, dV = \int_{\Omega} \frac{\partial}{\partial x_j} (v_i \mathbf{E}_{ji}) \, dV = \\
&= \int_{\Omega} \frac{\partial}{\partial x_i} (\mathbf{E}_{ij} v_j) \, dV = \int_{\Omega} \nabla \cdot (\mathbf{E}\mathbf{v}) \, dV
\end{aligned}$$

We need to explain what is the meaning of the integrand in the obtained volume integral. To see this, let us use the equivalent index form

$$\begin{aligned}
\nabla \cdot (\mathbf{E}\mathbf{v}) &\equiv \frac{\partial}{\partial x_i} (\mathbf{E}_{ij} v_j) = \left( \frac{\partial}{\partial x_i} \mathbf{E}_{ij} \right) v_j + \underbrace{\mathbf{E}_{ij} \left( \frac{\partial}{\partial x_i} v_j \right)}_{= \mathbf{E}_{ji} \left( \frac{\partial}{\partial x_i} v_j \right) = \mathbf{E}_{ij} \left( \frac{\partial}{\partial x_j} v_i \right)} = \text{Div}(\mathbf{E}) \cdot \mathbf{v} + \mathbf{E} : \nabla \mathbf{v}
\end{aligned}$$

The second term is a scalar referred to as the **scalar product of tensors**, in this case the stress tensor and the gradient of velocity (remind that since the velocity is the vector field, its gradient is the tensor field!). Moreover, the second terms can be transformed as follows

$$\mathbf{E} : \nabla \mathbf{v} = \mathbf{E} : (\mathbf{D} + \mathbf{R}) = \mathbf{E} : \mathbf{D}$$

where  $\mathbf{D}$  and  $\mathbf{R}$  are – respectively - the deformation and rotation rate tensors.

Indeed, the scalar product of the stress tensor  $\underline{\mathbf{E}}$  and the rotation tensor  $\mathbf{R}$  is zero, because the former is symmetric while the latter is antisymmetric

$$\underline{\mathbf{E}} : \mathbf{R} \equiv \frac{1}{2} \underline{\mathbf{E}}_{ij} \left( \frac{\partial}{\partial x_j} v_i - \frac{\partial}{\partial x_i} v_j \right) = \frac{1}{2} \underline{\mathbf{E}}_{ij} \frac{\partial}{\partial x_j} v_i - \frac{1}{2} \underline{\mathbf{E}}_{ij} \frac{\partial}{\partial x_i} v_j = \frac{1}{2} \underline{\mathbf{E}}_{ij} \frac{\partial}{\partial x_j} v_i - \frac{1}{2} \underline{\mathbf{E}}_{ji} \frac{\partial}{\partial x_j} v_i = 0$$

Thus, the power developed by the surface forces can be finally written as

$$P_S = \int_{\Omega} \text{Div}(\underline{\mathbf{E}}) \cdot \mathbf{v} dV + \int_{\Omega} \underline{\mathbf{E}} : \mathbf{D} dV$$

We will soon see that there exists a good reason for such splitting of this expression into two parts.

The other surface integral appears due to the heat flux through the boundary of the control volume

$$Q_{\partial\Omega} = - \int_{\partial\Omega} \mathbf{q}_h \cdot \mathbf{n} dS$$

where  $\mathbf{q}_h$  is the flux vector of the heat conducted by the fluid. In general, we can simply apply GGO theorem to obtain

$$Q_{\partial\Omega} = - \int_{\partial\Omega} \mathbf{q}_h \cdot \mathbf{n} dS = - \int_{\Omega} \nabla \cdot \mathbf{q}_h dV$$

In a general model of heat conduction, the vector  $\mathbf{q}_h$  is assumed to be equal

$$\mathbf{q}_h = -\mathbf{A}\nabla T$$

i.e., it is equal to the product of the heat conduction tensor  $\mathbf{A}$  (which characterized the medium's ability to conduct heat) and the gradient of temperature.

There are some constraints imposed on the properties of the tensor (matrix)  $\mathbf{A}$  which ensure that the heat always “travels” **from warmer to cooler parts** of the medium and the stream of heat does not “circulate”.

If the medium is isotropic, then the tensor is spherical and can be written as follows

$$\mathbf{A} = \lambda \mathbf{I}$$

The positive number  $\lambda$  is called the heat conduction coefficient of the medium. The heat flux is then expressed as

$$\mathbf{q}_h = -\lambda \nabla T \quad (\text{the Fourier law})$$

i.e., this vector is parallel to the gradient of temperature. The corresponding source term in the energy equation reads

$$Q_{\partial\Omega} = -\int_{\Omega} \nabla \cdot \mathbf{q}_h dV = \int_{\Omega} \nabla \cdot (\lambda \nabla T) dV$$

All terms in the energy equation can be now gathered in the single volume integral. The standard argument based on the arbitrariness of the control volume  $\Omega$  leads to the **differential equation of energy conservation**

$$\rho \frac{D}{Dt} e = \text{Div}(\mathbf{E}) \cdot \mathbf{v} + \mathbf{E} : \mathbf{D} + \rho \mathbf{f} \cdot \mathbf{v} + \rho \gamma_h + \nabla \cdot (\lambda \nabla T)$$

## INTERNAL ENERGY AND VISCOUS DISSIPATION

It is instructive to **consider separately the balance of kinetic and internal energy**. To this end, consider the linear momentum equation multiplied (in the sense of the inner product) by the velocity vector  $\boldsymbol{v}$

$$\rho \frac{D}{Dt} \boldsymbol{v} \cdot \boldsymbol{v} = \rho \boldsymbol{f} \cdot \boldsymbol{v} + \text{Div}(\boldsymbol{\Xi}) \cdot \boldsymbol{v}$$

Equivalently, we have

$$\rho \frac{D}{Dt} \left( \frac{1}{2} \boldsymbol{v}^2 \right) = \rho \boldsymbol{f} \cdot \boldsymbol{v} + \text{Div}(\boldsymbol{\Xi}) \cdot \boldsymbol{v}$$

..

In the next step, the above equation is integrated in the volume  $\Omega$  which yields the following integral expression for the temporal rate of change of the kinetic energy

$$\frac{d}{dt} \left\{ \int_{\Omega} \frac{1}{2} \rho \boldsymbol{v}^2 dV \right\}_{\text{production}} = \int_{\Omega} \rho \boldsymbol{f} \cdot \boldsymbol{v} dV + \int_{\Omega} \text{Div}(\boldsymbol{\Xi}) \cdot \boldsymbol{v} dV$$



Performing the transformation of the second right-hand side term, we arrive at the equivalent form

$$\frac{d}{dt} \left\{ \int_{\Omega} \frac{1}{2} \rho v^2 dV \right\} \text{production} = \int_{\Omega} \rho \mathbf{f} \cdot \mathbf{v} dV + \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{E} \mathbf{n} dS - \int_{\Omega} \mathbf{E} : \mathbf{D} dV$$

Using the general energy balance, we conclude that the net change rate of internal energy is

$$\frac{d}{dt} \underbrace{\int_{\Omega} \rho u dV}_U = \int_{\Omega} \rho q dV + \int_{\partial\Omega} \lambda \nabla T \cdot \mathbf{n} dS + \int_{\Omega} \mathbf{E} : \mathbf{D} dV$$

Observe that both formulae for different forms of energy contain the same term

$$\mathcal{T} := \int_{\Omega} \mathbf{E} : \mathbf{D} dV$$

but with **opposite signs!** Hence, this term describes the **transfer of mechanical energy into internal energy** (or vice versa).

Let's look at the structure of this term in details. The **stress tensor** for the linear fluids is expressed as

$$\mathbf{E}_{ij} = \left[ -p + \left( \zeta - \frac{2}{3} \mu \right) \frac{\partial}{\partial x_k} v_k \right] \delta_{ij} + 2\mu D_{ij}$$

Hence, we have

$$\begin{aligned} \mathbf{E} : \mathbf{D} &= \mathbf{E}_{ij} D_{ij} = -p \delta_{ij} D_{ij} + \left( \zeta - \frac{2}{3} \mu \right) \frac{\partial}{\partial x_k} v_k \delta_{ij} D_{ij} + 2\mu D_{ij} D_{ij} = \\ &= -p D_{ii} + \left( \zeta - \frac{2}{3} \mu \right) \frac{\partial}{\partial x_k} v_k D_{ii} + 2\mu D_{ij} D_{ij} = \\ &= \underbrace{-p \nabla \cdot \mathbf{v}}_{\substack{>0 \text{ if } \nabla \cdot \mathbf{v} < 0 \text{ (compression)} \\ <0 \text{ if } \nabla \cdot \mathbf{v} > 0 \text{ (expansion)}}} + \underbrace{\left( \zeta - \frac{2}{3} \mu \right) (\nabla \cdot \mathbf{v})^2 + 2\mu \overbrace{\mathbf{D} : \mathbf{D}}^{=tr \mathbf{D}^2}}}_{\text{positive-definite part}} \end{aligned}$$

The “energy transfer” term takes the following frame-invariant form

$$\int_{\Omega} \mathbf{E} : \mathbf{D} dV = \underbrace{-\int_{\Omega} p \nabla \cdot \mathbf{v} dV}_{\substack{\text{internal-to-mechanical} \\ \text{(if } <0 \text{) or reverse (if } >0 \text{)} \\ \text{energy transfer}}} + \underbrace{\left( \zeta - \frac{2}{3} \mu \right) \int_{\Omega} (\nabla \cdot \mathbf{v})^2 dV + 2\mu \int_{\Omega} \mathbf{D} : \mathbf{D} dV}_{\substack{\text{irreversible mechanical-to-internal energy transfer, i.e.} \\ \text{dissipation of the mechanical energy due to internal "friction"}}$$

For incompressible flows we have  $\nabla \cdot \mathbf{v} = 0$ , thus one obtains

$$\int_{\Omega} \mathbf{E} : \mathbf{D} dV = 2\mu \int_{\Omega} \mathbf{D} : \mathbf{D} dV = \mu \int_{\Omega} \frac{\partial}{\partial x_j} v_i \left( \frac{\partial}{\partial x_j} v_i + \frac{\partial}{\partial x_i} v_j \right) dV \equiv \mathcal{R}$$

The quantity  $\mathcal{R}$  is called the **dissipation rate**.

## FIRST INTEGRAL OF THE ENERGY EQUATION

If the fluid is ideal (i.e., inviscid and not heat-conducting) then **first integral of the energy equation** can be derived. The derivation procedure is similar as in the case of the **Bernoulli Equation**. The main difference is that **the flow need not to be barotropic** – we assume only **flow steadiness** and **potentiality of the volume force field**, i.e. that  $\mathbf{f} = \nabla\Phi$ .

We begin with the differential energy equation, which in the case of an ideal fluid reduces to

$$\rho \frac{D}{Dt} \left( u + \frac{1}{2} \mathbf{v}^2 \right) = -\nabla \cdot (p\mathbf{v}) + \rho \mathbf{f} \cdot \mathbf{v}$$

By expanding the pressure term, this equations can be re-written equivalently as

$$\rho \frac{D}{Dt} \left( u + \frac{1}{2} \mathbf{v}^2 \right) = -p \nabla \cdot \mathbf{v} - \mathbf{v} \cdot \nabla p + \rho \mathbf{f} \cdot \mathbf{v}$$

Since the volume force is potential, the corresponding term in the right-hand side can be transformed as follows

$$\rho \mathbf{f} \cdot \mathbf{v} = \rho \mathbf{v} \cdot \nabla \Phi = \rho (\underbrace{\partial_t \Phi}_{=0} + \mathbf{v} \cdot \nabla \Phi) = \rho \frac{D}{Dt} \Phi$$

Moreover, due to flow steadiness we have

$$\mathbf{v} \cdot \nabla p = \partial_t p + \mathbf{v} \cdot \nabla p = \frac{D}{Dt} p = 0$$

Next, from the mass conservation equation

$$\frac{D}{Dt} \rho + \rho \nabla \cdot \mathbf{v} = 0$$

we get the following expression for divergence of the velocity field

$$\nabla \cdot \mathbf{v} = -\frac{1}{\rho} \frac{D}{Dt} \rho$$

The energy equation can be now written in the following form

$$\frac{D}{Dt} \left( u + \frac{1}{2} \mathbf{v}^2 \right) = \underbrace{\frac{p}{\rho^2} \frac{D}{Dt} \rho - \frac{1}{\rho} \frac{D}{Dt} p}_{= -\frac{D}{Dt} (p/\rho)} + \frac{D}{Dt} \Phi$$

or

$$\frac{D}{Dt} \left( \underbrace{u + p/\rho}_= i + \frac{1}{2} \mathbf{v}^2 - \Phi \right) = 0$$

where  $i = u + p/\rho$  denotes the **mass-specific enthalpy** of the fluid.

Thus, the energy equation can be written as

$$\frac{D}{Dt} \left( i + \frac{1}{2} v^2 - \Phi \right) = 0$$

Since the flow is **stationary**, the above equation is equivalent to

$$\mathbf{v} \cdot \nabla \left( i + \frac{1}{2} v^2 - \Phi \right) = 0$$

Using the same arguments as in the case of the Bernoulli Eq., we conclude that **along each individual streamline**

$$i + \frac{1}{2} v^2 - \Phi = C_e = \text{const}$$

In particular, for the Clapeyron gas  $i = c_p T$  and we get

$$c_p T + \frac{1}{2} v^2 - \Phi = \text{const} \quad , \quad c_p = \frac{\kappa}{\kappa - 1} R$$

In general the energy constant  $C_e$  can be different for each streamline. If  $C_e$  is the same for all streamlines then the flow is called **homoenergetic**.

Let us recall that if the flow is **barotropic** then along each streamline we have

$$P + \frac{1}{2}v^2 - \Phi = C_B = \text{const}$$

Thus, when the flow is barotropic then

$$i - P = C_e - C_B = \text{const}$$

i.e., the enthalpy  $i$  and the pressure potential  $P$  differ only by an additive constant.

If additionally the fluid is incompressible then its internal energy  $u$  is fixed and the specific enthalpy can be defined as  $p / \rho$ . Thus, **in the incompressible case, the energy and Bernoulli equations are identical.**



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