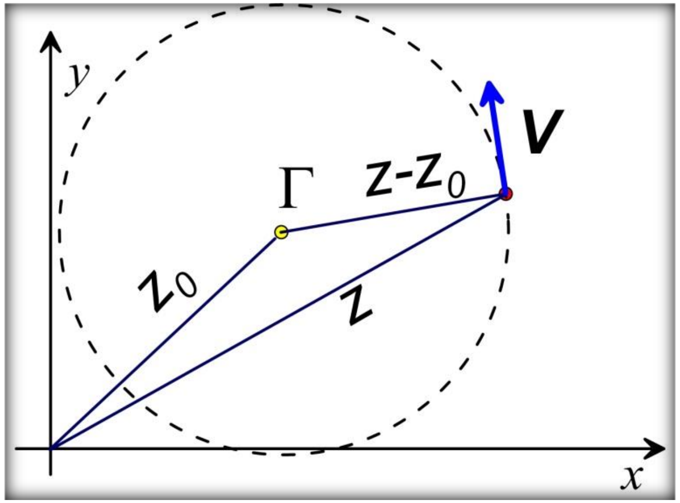
LECTURE 3

AERODYNAMICS OF A THIN AIRFOIL

AIM AND SCOPE:

- Presentation of the concept of potential flow modelling using singular spatially distributed vorticity carriers
- Application of this approach to the theory of a thin airfoil
- Derivation of formulae for aerodynamic force and moment generated by a thin airfoil
- Determination of the pressure and aerodynamic centers
- Application of the thin airfoil theory to the symmetric airfoil with (rear) flap.

1. Velocity induced by a vortex line



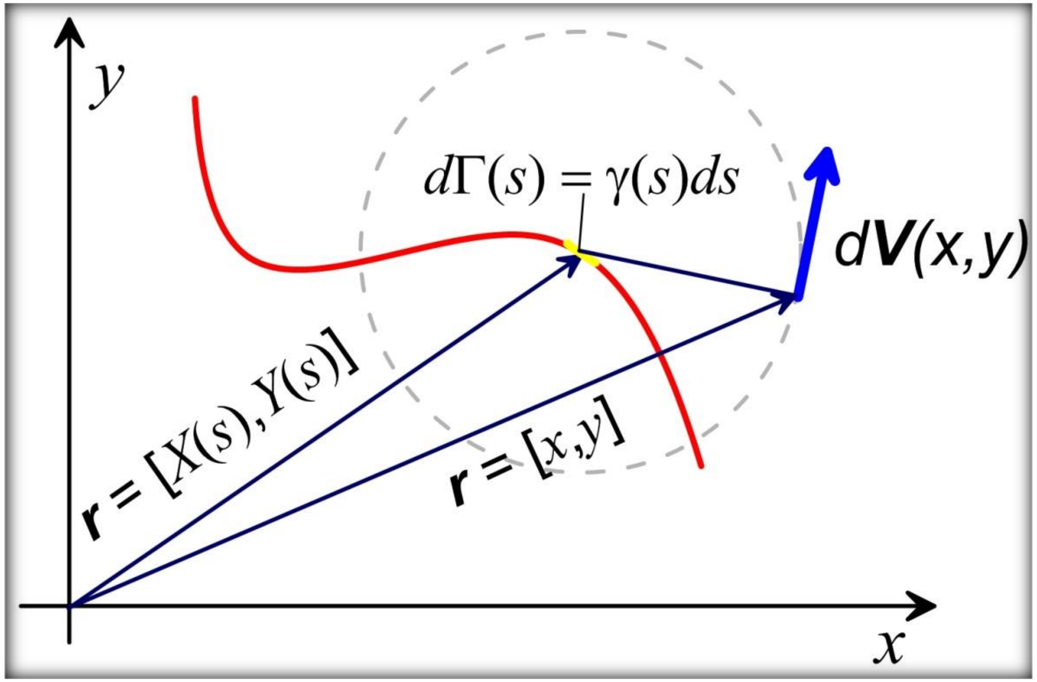
We have already introduced the (potential) point vortex. Let's remind its law of induction in the complex form (the center of the vortex is located at the point $z_0 = x_0 + iy_0$)

$$V_\Gamma(z) = -i\frac{\Gamma}{2\pi}\frac{1}{z-z_0} = -i\frac{\Gamma}{2\pi}\frac{\overline{z-z_0}}{|z-z_0|^2} = -i\frac{\Gamma}{2\pi}\frac{x-x_0-i(y-y_0)}{(x-x_0)^2+(y-y_0)^2} =$$

$$= -\frac{\Gamma}{2\pi}\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2} - i\frac{\Gamma}{2\pi}\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2} = u - i\upsilon$$

Hence, the Cartesian components of the induced velocity vector are ..

$$u_\Gamma = -\frac{\Gamma}{2\pi}\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2} \quad , \quad \upsilon_\Gamma = \frac{\Gamma}{2\pi}\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2}$$



Consider the velocity field induced by the circulation distributed continuously along the vortex line $x = X(s),\ y = Y(s)$.

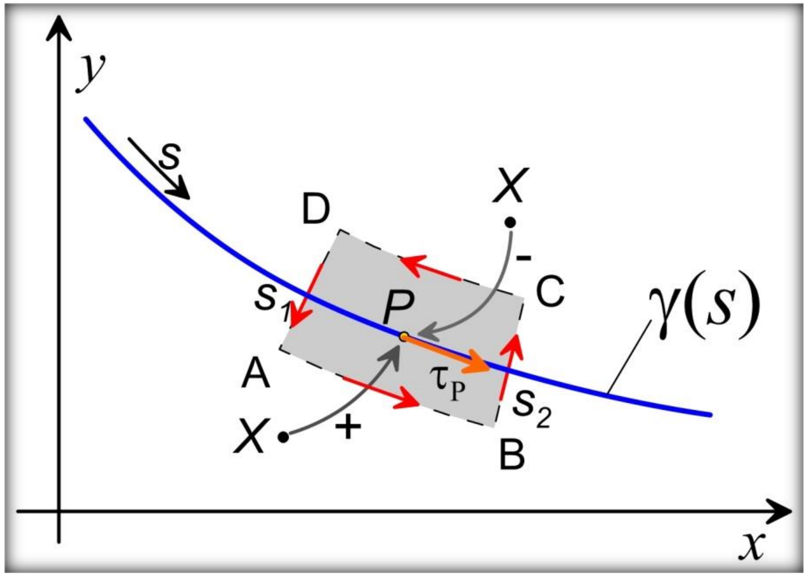
The linear density of the circulation along this line is $\gamma = \gamma(s)$.

The Cartesian components of the velocity field induced by this line are

$$u_\gamma = -\frac{1}{2\pi}\int_0^S \frac{\gamma(s)[y - Y(s)]}{[x - X(s)]^2 + [y - Y(s)]^2} ds \quad , \quad \upsilon_\gamma = \frac{1}{2\pi}\int_0^S \frac{\gamma(s)[x - X(s)]}{[x - X(s)]^2 + [y - Y(s)]^2} ds$$

Assume that the velocity induced by the VL at the point P (located on the VL) is finite. Then:

- Velocity normal to the line computed along any line crossing the VL at this point changes continuously.
- Velocity tangent to the VL computed along any line crossing the VL at the point P experiences a jump (meaning – it is discontinuous) equal to the value of the function γ at the point



The latter statement can be substantiated by means of the Stokes Theorem applied to the curvilinear quadrilateral ABCD and its contour

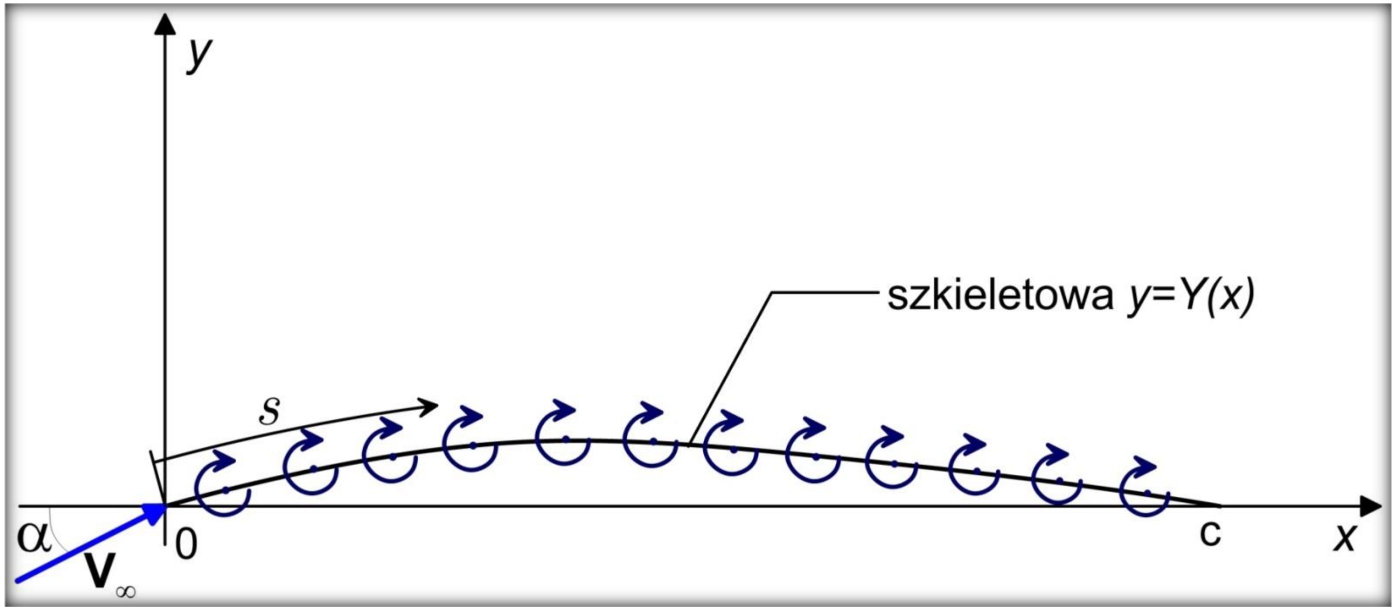
$$\oint_{ABCD} \boldsymbol{v}\cdot d\boldsymbol{s} = \int_{\boxed{ABCD}} \omega\, dxdy = \int_{s_1}^{s_2} \gamma(s)\, ds$$

Exercise: using the ST and continuity of the normal velocity across the VL, show that

$$\left(\lim_{X\to P^+} \boldsymbol{v} - \lim_{X\to P^-} \boldsymbol{v}\right)\cdot \tau_P = \gamma(s_P)$$

Vortex model od a thin airfoil

Variant 1



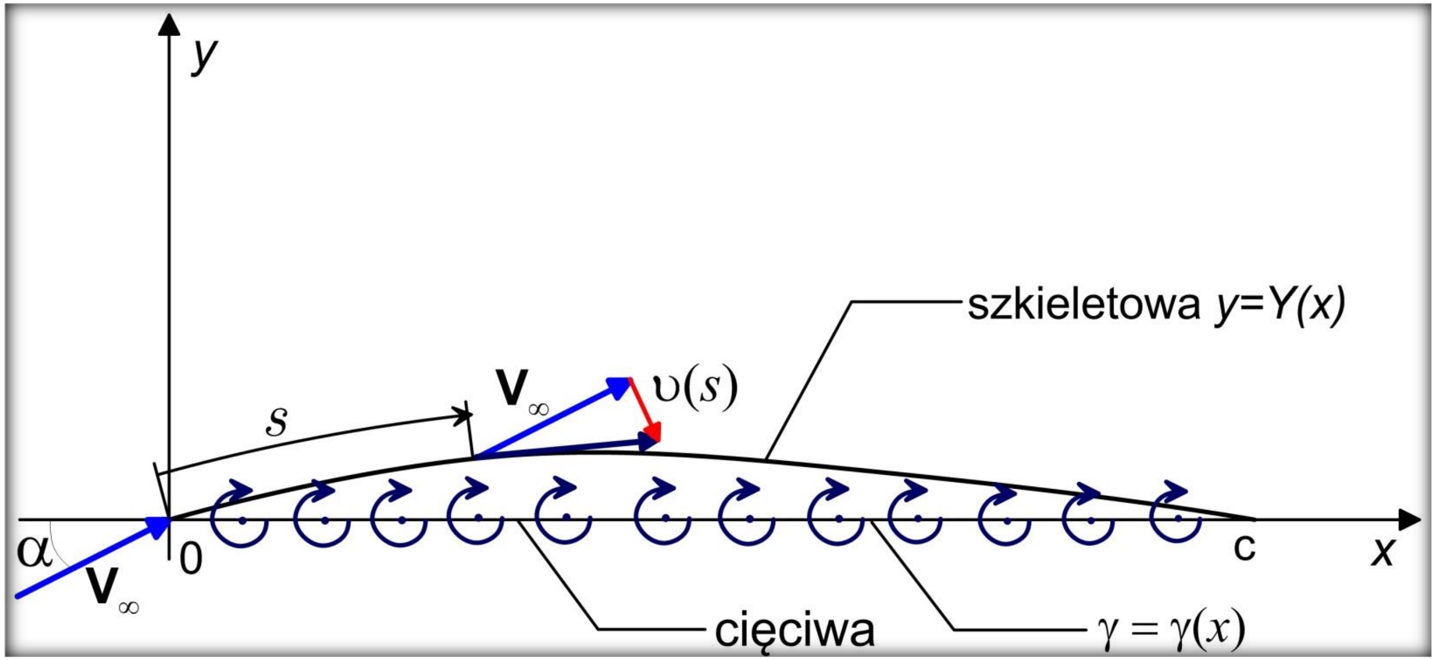
Vocabulary

Polish	English
szkieletowa	*camber line*

Flow past a thin airfoil is due to the superposition of the free-stream and the velocity induced by the vortex line shaped identically as the camber line. **The distribution of the circulation is such that the camber line is the streamline of this flow.**

But, if the vertical deflection of the camber line, i.e., the camber of the airfoil, is small, then …

Variant 2 (final)



Vocabulary

Polish	English
cięciwa	*chord*

- The circulation is distributed continuously along the chord line, in the interval $[0, c]$.
- We assume small values of the camber, hence the velocity induced at any point $[x, Y(x)]$ lying on the camber line does not significantly differ from the velocity induced at the point $[x, 0]$.

Consequently, the normal velocity induced along the camber line can be expressed by the approximate formula

$$w_{\gamma,n}[x,Y(x)] \approx u_\gamma[x,0]n_x[x,Y(x)] + \upsilon_\gamma[x,0]n_y[x,Y(x)]$$

The unary vector normal to the camber line at the point $[x,Y(x)]$ is

$$n_x = \frac{-Y'(x)}{\sqrt{1+[Y'(x)]^2}} \approx -Y'(x) \quad , \quad n_y = \frac{1}{\sqrt{1+[Y'(x)]^2}} \approx 1$$

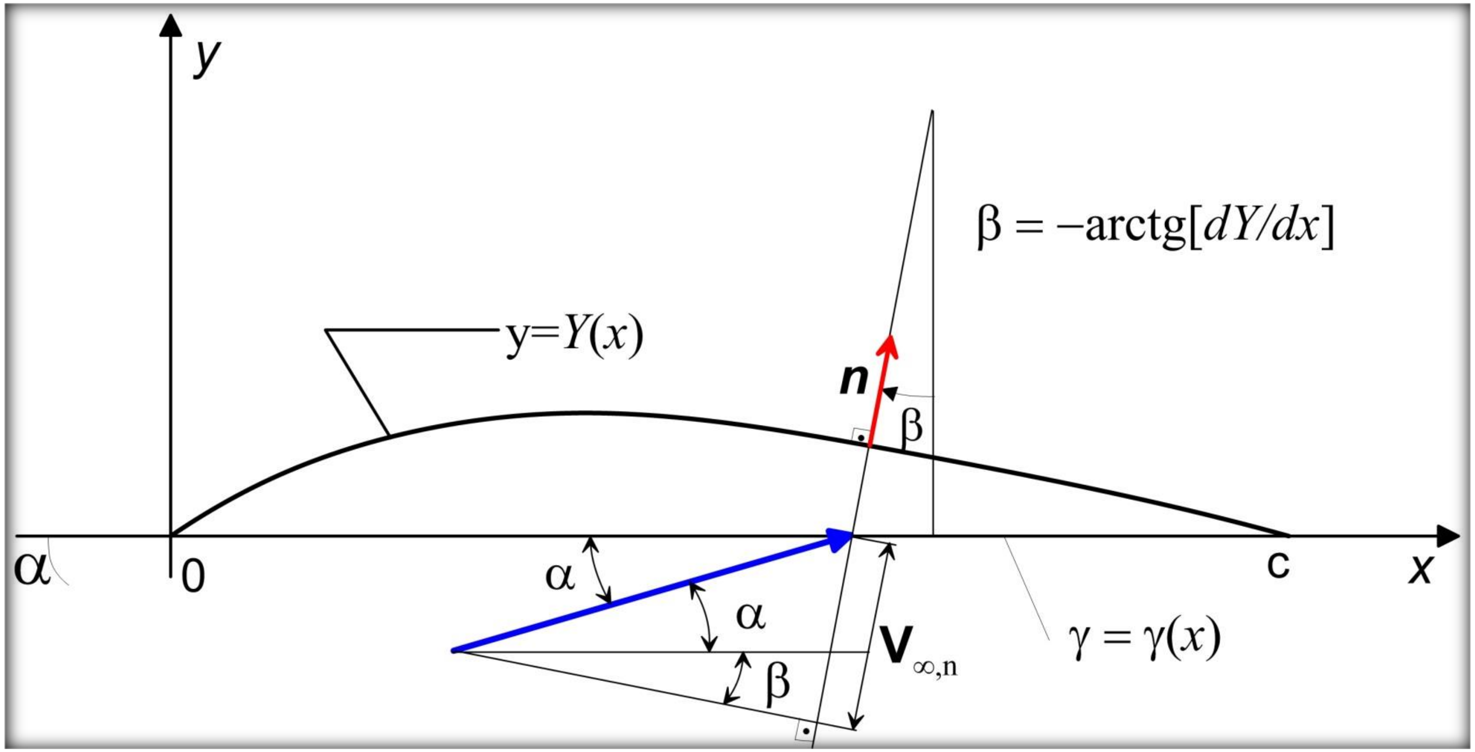
Hence

$$w_{\gamma,n}[x,Y(x)] \approx -u_\gamma[x,0]n_x[x,Y(x)] + \upsilon_\gamma[x,0]n_y[x,Y(x)] \approx$$

$$\approx u_\gamma[x,0]Y'(x) + \upsilon_\gamma[x,0] \approx \upsilon_\gamma[x,0] = \frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)d\xi}{x-\xi}$$

The condition of airfoil impermeability – a complete normal velocity at the camber line is zero

$$\boxed{V_{\infty,n} + w_{\gamma,n} = 0}$$



It follows form the figure that

$$V_{\infty,n} = V_\infty \sin(\alpha+\beta) \approx V_\infty(\alpha+\beta) = V_\infty\{\alpha - arctg[Y'(x)]\} \approx V_\infty[\alpha - Y'(x)]$$

Impermeability condition takes the form (**basic equation of the this airfoil theory**)

$$\boxed{V_\infty[\alpha - Y'(x)] + \frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)\,d\xi}{x-\xi} = 0}$$

We introduce the coordinate transformation …

$$\xi = \tfrac{1}{2}c(1-\cos\theta) \quad , \quad x = \tfrac{1}{2}c(1-\cos\theta_0)$$

Then $d\xi = \tfrac{1}{2}c\sin\theta$ and $\begin{cases} \xi = 0 \Rightarrow 1-\cos\theta = 0 \Rightarrow \theta = 0 & \textit{leading edge} \\ \xi = c \Rightarrow 1-\cos\theta = 2 \Rightarrow \theta = \pi & \textit{trailing edge} \end{cases}$

Consider **thin symmetric airfoil** (within this theory – equivalent to the flat, infinitely thin plate), i.e., assume that $Y'(x) \equiv 0$.

The basic equation takes the following form

$$\frac{1}{2\pi}\int_0^{\pi} \frac{\gamma(\theta)\sin\theta d\theta}{\cos\theta - \cos\theta_0} = -V_\infty \alpha$$

From the properties of the velocity field induced by the vortex, it follows that we must impose the additional condition

$$\gamma(\pi) = 0$$

Otherwise, the velocity at the trailing edge will be ambiguous – the Kutta-Joukovsky condition will be violated!

In order to find the solution, the following Glauert integrals are needed

$$\int_0^{\pi} \frac{\cos n\theta}{\cos\theta - \cos\theta_0} d\theta = \pi \frac{\sin n\theta_0}{\sin\theta_0} \quad , \quad n = 0, 1, 2, \ldots$$

$$\int_0^{\pi} \frac{\sin n\theta \sin\theta}{\cos\theta - \cos\theta_0} d\theta = -\pi \cos n\theta_0 \quad , \quad n = 0, 1, 2, \ldots$$

In particular …
$$\int_0^{\pi} \frac{\cos\theta}{\cos\theta - \cos\theta_0} d\theta = \pi$$

Consider the function
$$\gamma_1(\theta) = K \cos\theta / \sin\theta$$

We have
$$\frac{1}{2\pi} \int_0^{\pi} \frac{\gamma_1(\theta) \sin\theta d\theta}{\cos\theta - \cos\theta_0} = \frac{K}{2\pi} \int_0^{\pi} \frac{\cos\theta d\theta}{\cos\theta - \cos\theta_0} = \frac{K}{2}$$

If the constant K is such that $\frac{1}{2} K = -V_\infty \alpha \quad \Rightarrow \quad K = -2V_\infty \alpha$ then the function

$$\gamma_1(\theta) = -2V_\infty \alpha \cos\theta / \sin\theta = -2V_\infty \alpha \cot\theta$$

fulfils the basic equation! **Unfortunately,** $\gamma_1(\theta)$ **does not satisfy the K-J condition!**

Due to linearity, we can modify our solution by adding any component $\gamma_2(\theta)$ such that

$$\int_0^{\pi} \frac{\gamma_2(\theta)\sin\theta d\theta}{\cos\theta - \cos\theta_0} = 0$$

First type of the Glauert integral for $n = 0$ yields

$$\int_0^{\pi} \frac{1}{\cos\theta - \cos\theta_0} d\theta = 0$$

Hence, we can adopt $\gamma_2(\theta) = \dfrac{K}{\sin\theta}$.

Full distribution of the circulation of the vortex line is

$$\gamma(\theta) = -2V_{\infty}\alpha \frac{\cos\theta}{\sin\theta} + \frac{K}{\sin\theta}$$

Note that (exercise!) the Kutta-Joukovsky condition is satisfied iff $K = -2V_{\infty}\alpha$.

Finally

$$\gamma(\theta) = -2V_\infty \alpha \frac{\cos\theta + 1}{\sin\theta} = -2V_\infty \alpha \frac{2\cos^2(\frac{1}{2}\theta)}{2\sin(\frac{1}{2}\theta)\cos(\frac{1}{2}\theta)} = -2V_\infty \alpha \cot(\tfrac{1}{2}\theta)$$

Note that the density of circulation at the leading edge ($\theta = 0$) is infinite!!!

Let us calculate the total charge of circulation of the vortex line. In spite of the leading-edge singularity, this charge is finite.

Indeed, we have

$$\Gamma = \int_0^c \gamma(\xi)\,d\xi = -2V_\infty \alpha \int_0^\pi \frac{1+\cos\theta}{\cancel{\sin\theta}} \frac{c}{2} \cancel{\sin\theta}\, d\theta = -V_\infty c\alpha \int_0^\pi (1+\cos\theta)\,d\theta = -\pi\alpha V_\infty c$$

Aerodynamic force and moment

From the Kutta-Joukovsky formula one obtains the aerodynamic force vector (only lift component exist!)

$$\boldsymbol{L} = \underbrace{\rho V_\infty |\Gamma|}_{L} (-\sin\alpha\, \boldsymbol{e}_x + \cos\alpha\, \boldsymbol{e}_y)$$

The value of the lift force is

$$\boxed{L = \pi\alpha\rho V_\infty^2 c = C_L \underbrace{\tfrac{1}{2}\rho V_\infty^2}_{q_\infty} c = C_L q_\infty c}$$

Lift force coefficient $$C_L = 2\pi\alpha$$

The slope of the lift force characteristics $C_L = C_L(\alpha)$ is equal $$\frac{dC_L}{d\alpha} = 2\pi$$

Let us calculate the aerodynamic moment with respect to the leading edge

For small angles of attack, this moment can be calculated as follows

$$M_0 \approx \int_0^c x\,dL(x) \approx \int_0^c x\,L'(x)\,dx = -\rho V_\infty \int_0^c x\,\Gamma'(x)\,dx = -\rho V_\infty \int_0^c x\,\gamma(x)\,dx =$$

$$= -\rho V_\infty \int_0^\pi \tfrac{1}{2}c(1-\cos\theta)(-2V_\infty\alpha)\frac{1+\cos\theta}{\sin\theta}\tfrac{1}{2}c\sin\theta\,d\theta = \tfrac{1}{2}\alpha\rho V_\infty^2 c^2 \int_0^\pi \sin^2\theta\,d\theta =$$

$$= \tfrac{1}{2}\pi\alpha\,\tfrac{1}{2}\rho V_\infty^2 c^2 = \tfrac{1}{4}c\,\pi\alpha\,\rho V_\infty^2 c = \tfrac{1}{4}c\,L$$

We conclude that – for small angles of attack – the point where the aerodynamic (lift) force is applied – called the **<u>center of pressure</u> - is located at the distance ¼ c (a quarter of the chord length) from the leading edge.**

<u>Remark:</u> It is common convention in aircraft aerodynamics that the positive aerodynamic moment works towards increment of the airfoil angle of attack (hence, it makes the airfoil to turn clockwise). Thus, the moment acting on the symmetric airfoil at a positive angle of attack is negative.

We will adjust the sign of the moment to this convention

$$M_0 = -\tfrac{1}{4} c L$$

The moment coefficient is defined as

$$C_{m,0} = \frac{M_0}{q_\infty c^2} = -\frac{\tfrac{1}{4} c L}{q_\infty c^2} = -\frac{1}{4}\frac{L}{q_\infty c} = -\frac{1}{4} C_L$$

Note that the moment with respect to an arbitrary point P such that $x = x_P$ is equal

$$M_P = -\int_0^c (x - x_P)\, dL(x) = \underbrace{-\int_0^c x\, dL(x)}_{M_0} + x_P L = (-\tfrac{1}{4} c + x_P) L$$

In terms of the aerodynamic coefficients …

$$C_{m,P} = \frac{(-\tfrac{1}{4} c + x_P) L}{q_\infty c^2} = (-\frac{1}{4} + \bar{x}_P) C_L = C_{m,0} + \bar{x}_P C_L$$

If in particular $\overline{x}_P = \frac{1}{4}$, then we have $C_{m,c/4} = 0$.

One can see that the coefficient of the aerodynamic moment with respect to the center of pressure $\overline{x}_P = \frac{1}{4}$ is identically zero and hence it does not depend on the angle of attack (at least for sufficiently small angles), i.e.,

$$\frac{dC_{m,c/4}}{d\alpha} = 0$$

According to the definition, such point is called **the aerodynamic center**. Thus, **<u>for thin symmetric airfoils the aerodynamic center coincides with the center of pressure</u>**.

We will now extend our consideration **to non-symmetric thin airfoils**.

Let us remind the basic equation of the thin airfoil theory …

$$V_\infty[\alpha - Y'(x)] + \frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)\,d\xi}{x-\xi} = 0$$

which – after the coordinate transformation assumes the following form

$$\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\sin\theta\,d\theta}{\cos\theta - \cos\theta_0} = -V_\infty\alpha + V_\infty Y'[x(\theta_0)]$$

Solution of this equation in the non-symmetric case is more complex. We seek the distribution of the circulation along the chord line in the form of the **Fourier series**

$$\gamma(\theta) = -2V_\infty\left[A_0\frac{\cos\theta+1}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin n\theta\right]$$

After insertion to the equation, one obtains

$$\frac{A_0}{\pi}\int_0^{\pi}\frac{\cos\theta+1}{\cos\theta-\cos\theta_0}d\theta+\frac{1}{\pi}\sum_{n=1}^{\infty}A_n\int_0^{\pi}\frac{\sin n\theta\sin\theta}{\cos\theta-\cos\theta_0}d\theta=\alpha-Y'[x(\theta_0)]$$

Using again the Glauert integrals

$$\int_0^{\pi}\frac{\sin n\theta\sin\theta}{\cos\theta-\cos\theta_0}d\theta=-\pi\cos n\theta_0$$

$$\int_0^{\pi}\frac{\cos\theta+1}{\cos\theta-\cos\theta_0}d\theta=\underbrace{\int_0^{\pi}\frac{1}{\cos\theta-\cos\theta_0}d\theta}_{=0}+\underbrace{\int_0^{\pi}\frac{\cos\theta}{\cos\theta-\cos\theta_0}d\theta}_{\substack{=\pi\\ \text{1st Glauert integral}\\ \text{for } n=1}}=\pi$$

the above equation is transformed to the form

$$A_0-\sum_{n=1}^{\infty}A_n\cos n\theta_0=\alpha-Y'[x(\theta_0)]$$

Equivalently

$$Y'[x(\theta_0)] = (\alpha - A_0) + \sum_{n=1}^{\infty} A_n \cos n\theta_0$$

In order to find the Fourier coefficients $A_j,\ j = 0, 1, 2, \ldots$ one has to:

- express the function $Y' = Y'(x)$ by the transformed coordinate $\theta \in [0, \pi]$,
- calculate Fourier coefficients of the obtained function.

Let us denote $P(\theta_0) = Y'[x(\theta_0)]$. We expand the function $P(\theta_0)$ in a trigonometric series

$$P(\theta_0) = B_0 + \sum_{n=1}^{\infty} B_n \cos n\theta_0$$

In follows from the analysis that

$$B_0 = \tfrac{1}{\pi} \int_0^{\pi} P(\theta) d\theta \quad , \quad B_n = \tfrac{2}{\pi} \int_0^{\pi} P(\theta) \cos n\theta \, d\theta$$

Hence, we obtain the following relations

$$\alpha - A_0 = B_0 = \frac{1}{\pi}\int_0^{\pi} P(\theta)d\theta \quad \Rightarrow \quad A_0 = \alpha - \frac{1}{\pi}\int_0^{\pi} P(\theta)d\theta$$

$$A_n = B_n = \frac{2}{\pi}\int_0^{\pi} P(\theta)\cos n\theta\, d\theta$$

The corresponding distribution of the circulation is expressed by the formula

$$\gamma(\theta) = -2V_\infty\left[A_0\frac{\cos\theta + 1}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin n\theta\right]$$

Note that **the Kutta-Joukovsky condition is automatically fulfilled as** $\gamma(\pi) = 0$**.**

Total circulation connected to the thin airfoil is equal

$$\Gamma = \int_0^c \gamma(\xi)d\xi = \frac{c}{2}\int_0^{\pi}\gamma(\theta)\sin\theta d\theta = -cV_\infty\left[A_0\int_0^{\pi}(1+\cos\theta)d\theta + \sum_{n=1}^{\infty} A_n\int_0^{\pi}\sin n\theta\sin\theta d\theta\right]$$

Since

$$\int_0^{\pi}(1+\cos\theta)d\theta=\pi \qquad , \qquad \int_0^{\pi}\sin n\theta\sin\theta d\theta=\begin{cases}\pi/2 & for \quad n=1\\ 0 & for \quad n\neq 1\end{cases}$$

then

$$\boxed{\Gamma=-\pi cV_{\infty}(A_0+\tfrac{1}{2}A_1)}$$

Lift force L and lift coefficient C_L are

$$L=\rho V_{\infty}\left|\Gamma\right|=2\pi(A_0+\tfrac{1}{2}A_1)\tfrac{1}{2}\rho V_{\infty}^2c=2\pi(A_0+\tfrac{1}{2}A_1)q_{\infty}c$$

$$C_L=2\pi(A_0+\tfrac{1}{2}A_1)$$

Explicitly …

$$\boxed{C_L=2\pi\left[\alpha+\frac{1}{\pi}\int_0^{\pi}P(\theta)(\cos\theta-1)d\theta\right]}$$

We see that the slope $\dfrac{dC_L}{d\alpha}=2\pi$, i.e., **it does not depend on the airfoil camber**.

<u>Comment:</u>

In the Lecture 2, we have asked the Reader to demonstrate that the slope of the lift force characteristics for the Joukovsky's non-symmetric airfoil with zero thickness is expressed by the approximate formula $\frac{dC_L}{d\alpha} \approx 2\pi(1+2\bar{f}^{\,2})$. The small correction – proportional to the square of the camber – appears. However, the theory of a thin airfoil does not see this correction – this theory is sensitive to only "1st-order" effects. Can you explain why?

The formula for the lift force coefficient can be written as follows

$$C_L = 2\pi(\alpha - \alpha_0)$$

where
$$\alpha_0 = -\tfrac{1}{\pi}\int_0^{\pi} P(\theta)(\cos\theta - 1)\,d\theta$$

is **the (negative) angle of attack, at which the cambered airfoil is not producing any lift**. It follows also that

$$C_L(\alpha = 0) = 2\int_0^{\pi} P(\theta)(\cos\theta - 1)\,d\theta$$

Let us again determine the aerodynamic moment with respect to the leading edge

$$M_0 = -\rho V_\infty \int_0^c x\gamma(x)\,dx = -\rho V_\infty \int_0^c x\gamma(x)\,dx = -\tfrac{1}{4}\rho V_\infty c^2 \int_0^\pi (1-\cos\theta)\gamma(\theta)\sin\theta d\theta =$$

$$= \tfrac{1}{2}\rho V_\infty^2 c^2 \int_0^\pi (1-\cos\theta)\left[A_0 \frac{\cos\theta+1}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin n\theta\right]\sin\theta d\theta =$$

$$= \tfrac{1}{2}\rho V_\infty^2 c^2 \left[A_0 \int_0^\pi (1-\cos^2\theta)d\theta + \sum_{n=1}^{\infty} A_n \int_0^\pi (1-\cos\theta)\sin n\theta \sin\theta\, d\theta\right]$$

We need to evaluate the following integrals

$$\int_0^\pi (1-\cos^2\theta)d\theta = \int_0^\pi \sin^2\theta d\theta = \tfrac{1}{2}\pi$$

$$\int_0^\pi (1-\cos\theta)\sin n\theta \sin\theta\, d\theta = \int_0^\pi \sin n\theta \sin\theta\, d\theta - \tfrac{1}{2}\int_0^\pi \sin n\theta \sin 2\theta\, d\theta = \begin{cases} \pi/2 & for \quad n=1 \\ -\pi/4 & for \quad n=2 \\ 0 & for \quad n \notin \{1,2\} \end{cases}$$

Thus, we obtain the formula

$$M_0 = \tfrac{1}{4}\pi\rho V_\infty^2 c^2 (A_0 + A_1 - \tfrac{1}{2}A_2)$$

Adjusting sign to the standard aerodynamic convention, one has

$$M_0 = -\tfrac{1}{4}\pi\rho V_\infty^2 c^2 (A_0 + A_1 - \tfrac{1}{2}A_2) = -\tfrac{1}{2}\pi(A_0 + A_1 - \tfrac{1}{2}A_2) q_\infty c^2$$

The moment coefficient is equal

$$C_{m,0} = -\tfrac{1}{2}\pi(A_0 + A_1 - \tfrac{1}{2}A_2) = -\tfrac{1}{4}[C_L + \pi(A_1 - A_2)]$$

We have shown earlier that

$$C_{m,c/4} = C_{m,0} + \tfrac{1}{4}C_L = -\frac{\pi}{4}(A_1 - A_2) = \frac{\pi}{4}(A_2 - A_1)$$

The obtained value of $C_{m,c/4}$ does not depend of the angle of attack**. Hence, the point $x = \tfrac{1}{4}c$ is still the aerodynamic center, although it is not the center of pressure (as, in general, $C_{m,c/4} \neq 0$)**

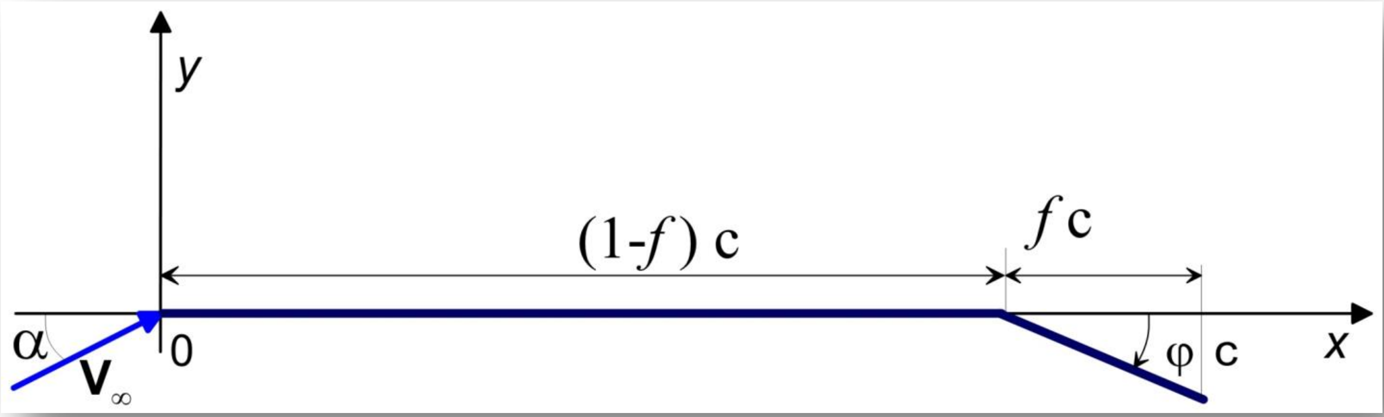
The actual position of the center of pressure x_P is

$$x_P = \frac{\frac{1}{2}\pi q_\infty c^2 (A_0 + A_1 - \frac{1}{2}A_2)}{2\pi(A_0 + \frac{1}{2}A_1)q_\infty c} = \frac{\frac{1}{2}\pi c[A_0 + \frac{1}{2}A_1 + \frac{1}{2}(A_1 - A_2)]}{\underbrace{2\pi(A_0 + \frac{1}{2}A_1)}_{C_L}} =$$

$$= \frac{c}{4} + \frac{c}{4}\frac{\pi}{C_L}(A_1 - A_2) = \frac{c}{4}\left[1 + \frac{\pi}{C_L}(A_1 - A_2)\right]$$

Thin symmetric airfoil with the flap

We will discussed shortly the model of a thin symmetric airfoil with the (rear) flap



Camber line and its derivative

$$Y(x)=\begin{cases} 0 & for \;\; x\in[0,x_f] \\ -\tan\varphi(x-x_f) & for \; x\in(x_f,c] \end{cases} \qquad , \qquad x_f=(1-f)c$$

$$Y'(x)=\begin{cases} 0 & for \; x\in[0,x_f) \\ -\tan\varphi & for \; x\in[x_f,c] \end{cases}$$

As before, we apply the coordinate transformation $x \to \theta$.

We have tp determine the angular coordinate corresponding to the flap hinge

$$x_f = (1-f)c \;\Rightarrow\; \tfrac{1}{2}c(1-\cos\theta_f) = (1-f)c$$

Hence
$$\cos\theta_f = 2f - 1$$

For instance, if $f = 0.15$ then $\theta_f = \mathrm{acos}(-0.7) \approx 134.43^0$.

The function $P(\theta)$ is defined by the formula

$$P(\theta) = Y'[\tfrac{1}{2}c(1-\cos\theta)] = \begin{cases} 0 \quad \mathit{for} \;\; \theta \in [0, \theta_f) \\ -\tan\varphi \;\; \mathit{for} \; \theta \in [\theta_f, \pi] \end{cases}$$

We have to calculate three first Fourier coefficients …

$$\frac{1}{\pi}\int_0^\pi P(\theta)d\theta=\frac{1}{\pi}\int_{\theta_f}^\pi(-tg\varphi)d\theta=-\left[1-\frac{\theta_f}{\pi}\right]tg\varphi$$

Hence $$A_0=\alpha+(1-\theta_f/\pi)tg\varphi$$

Next …

$$A_1=\frac{2}{\pi}\int_0^\pi P(\theta)\cos\theta d\theta=\frac{2}{\pi}(-tg\varphi)\int_{\theta_f}^\pi\cos\theta d\theta=\frac{2}{\pi}(-tg\varphi)(-\sin\theta_f)=\frac{2}{\pi}\sin\theta_f tg\varphi$$

$$A_2=\frac{2}{\pi}\int_0^\pi P(\theta)\cos 2\theta d\theta=\frac{2}{\pi}(-tg\varphi)\int_{\theta_f}^\pi\cos 2\theta d\theta=\frac{1}{\pi}(-tg\varphi)(-\sin 2\theta_f)=\frac{1}{\pi}\sin 2\theta_f tg\varphi$$

The lift coefficient of the this symmetric airfoil with the flap is expressed as follows

$$C_L = 2\pi(A_0 + \tfrac{1}{2}A_1) = 2\pi\alpha + 2\pi(1 - \theta_f/\pi + \tfrac{1}{\pi}\sin\theta_f)tg\varphi =$$
$$\approx 2\pi\alpha + 2(\pi - \theta_f + \sin\theta_f)\varphi$$

We have shown earlier that the moment coefficient is equal

$$C_{m,0} = -\tfrac{1}{2}\pi(A_0 + A_1 - \tfrac{1}{2}A_2)$$

Upon insertion, we have the formula

$$C_{m,0} = -\tfrac{\pi}{2}\alpha - \tfrac{1}{2}[\pi - \theta_f + (2 - \cos\theta_f)\sin\theta_f]\varphi$$

Let us take a look at the numbers. Assume again 15% flap (i.e., $f = 0.15$). We already know that $\theta_f \approx 134.4^0$. For this data we obtain

$$2(\pi - \theta_f + \sin\theta_f) \approx 3.02$$

$$\tfrac{1}{2}[\pi - \theta_f + (2 - \cos\theta_f)\sin\theta_f] \approx 1.36$$

Thus, we have obtained the following relations

$$C_L = 2\pi\alpha + 3.02\varphi \quad , \quad C_{m,0} = -\tfrac{\pi}{2}\alpha - 1.36\varphi$$

The moment coefficient $C_{m,c/4}$ is equal

$$C_{m,c/4} = \tfrac{\pi}{4}(A_2 - A_1) = \tfrac{\pi}{4}\left(\tfrac{1}{\pi}\sin 2\theta_f tg\varphi - \tfrac{2}{\pi}\sin\theta_f tg\varphi\right) =$$
$$\approx \tfrac{1}{4}(\sin 2\theta_f - \sin\theta_f)\varphi$$

For the 15% flap $$C_{m,c/4} \approx -0.395\varphi$$

Exercise: Perform analogous analysis for the thin symmetric airfoil with the short flap at the leading angle (the slot). Assume that the hinge of the slot is located at the distance equal $f \cdot c$ from the leading edge. Compare the effects of 10% slot and 15% rear flap on the aerodynamic coefficients.