PROGRAM ROZWOJOWY
POLITECHNIKI WARSZAWSKIEJ

LECTURE 6
KINEMATICS OF FLUIDS –PART 2

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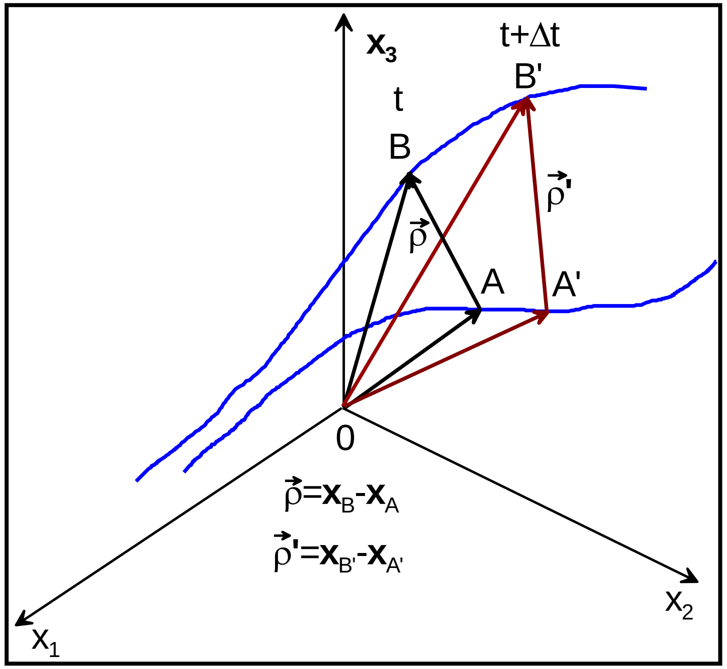
RELATIVE MOTION OF FLUID ELEMENTS

Consider two fluid elements located instantaneously at the close points **A** and **B**. We ask what happens to the relative position of these fluid elements after a short time interval Δt.



The location of the first fluid element after the time Δt can be expressed as follows

$$\boldsymbol{x}_{A'} = \boldsymbol{x}_A + \boldsymbol{v}(t, \boldsymbol{x}_A)\Delta t + O(\Delta t^2)$$

Since $\boldsymbol{x}_B = \boldsymbol{x}_A + \boldsymbol{\rho}$ then analogously we have

$$\boldsymbol{x}_{B'} = \boldsymbol{x}_A + \boldsymbol{\rho} + \boldsymbol{v}(t, \boldsymbol{x}_A + \boldsymbol{\rho})\Delta t + O(\Delta t^2),$$

where the vector $\boldsymbol{\rho}$ describes the relative position of the fluid elements at the time t.

During a short time interval Δt this vector has changed and can be expressed as

$$\boldsymbol{\rho}(t + \Delta t) = \boldsymbol{x}_{B'} - \boldsymbol{x}_{A'} = \boldsymbol{\rho}(t) + [\boldsymbol{v}(t, \boldsymbol{x}_A + \boldsymbol{\rho}) - \boldsymbol{v}(t, \boldsymbol{x}_A)]\Delta t + O(\Delta t^2) =$$

$$= \boldsymbol{\rho}(t) + \left[\nabla \boldsymbol{v}(t, \boldsymbol{x})\boldsymbol{\rho}\right]\Delta t + O(\Delta t^2, \Delta t\,|\boldsymbol{\rho}|^2)$$

In the above, we have dropped the lower index “A” at the location vector corresponding to the first element.

The rate of change of the vector describing the relative position of two close fluid elements can be calculated

$$\frac{d\boldsymbol{\rho}}{dt} = \lim_{\Delta t \to 0} \frac{\boldsymbol{\rho}(t+\Delta t) - \boldsymbol{\rho}(t)}{\Delta t} = \nabla \boldsymbol{\upsilon}(t, \boldsymbol{x})\,\boldsymbol{\rho} + O(|\,\boldsymbol{\rho}\,|^2).$$

We have introduced the matrix (tensor) called the **velocity gradient** $\left[\nabla \boldsymbol{\upsilon}\right]_{ij} = \frac{\partial}{\partial x_j}\upsilon_i$.

The velocity gradient $\nabla \boldsymbol{\upsilon}$ can be written as a sum of two tensors $\nabla \boldsymbol{\upsilon} = \boldsymbol{D} + \boldsymbol{R}$, where

$$\boldsymbol{D} = \tfrac{1}{2}[\nabla \boldsymbol{\upsilon} + (\nabla \boldsymbol{\upsilon})^T] \quad \text{or} \quad d_{ij} = \frac{1}{2}\left(\frac{\partial \upsilon_i}{\partial x_j} + \frac{\partial \upsilon_j}{\partial x_i}\right)$$ - symmetric tensor,

and

$$\boldsymbol{R} = \tfrac{1}{2}[\nabla \boldsymbol{\upsilon} - (\nabla \boldsymbol{\upsilon})^T] \quad \text{or} \quad r_{ij} = \frac{1}{2}\left(\frac{\partial \upsilon_i}{\partial x_j} - \frac{\partial \upsilon_j}{\partial x_i}\right)$$ - skew-symmetric tensor

We will show that **the change of the relative position of the fluid elements due to the action of the antysymmetric tensor $\boldsymbol{R}$ corresponds to the local "rigid" rotation of the fluid**.

Next, we will show that the action of the symmetric part $\boldsymbol{D}$ **corresponds to the "real" deformation, i.e. it is responsible of the change in shape and volume**.

To this end, we note that $r_{ij} = -\frac{1}{2} \in_{ijk} \omega_k$, where ω_k are the Cartesian components of the **vorticity** vector

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{\upsilon} = \underbrace{\in_{k\beta\gamma} \frac{\partial \upsilon_\gamma}{\partial x_\beta}}_{\omega_k} \boldsymbol{e}_k .$$

Indeed, we have

$$-\tfrac{1}{2} \in_{ijk} \in_{k\beta\gamma} \frac{\partial \upsilon_\gamma}{\partial x_\beta} = -\tfrac{1}{2} (\delta_{i\beta}\delta_{j\gamma} - \delta_{i\gamma}\delta_{j\beta}) \frac{\partial \upsilon_\gamma}{\partial x_\beta} = -\frac{1}{2}\left(\frac{\partial \upsilon_j}{\partial x_i} - \frac{\partial \upsilon_i}{\partial x_j} \right) =$$

$$= \frac{1}{2}\left(\frac{\partial \upsilon_i}{\partial x_j} - \frac{\partial \upsilon_j}{\partial x_i} \right) = r_{ij}$$

Thus, we can write

$$\boldsymbol{R}\boldsymbol{\rho} = r_{ij}\, \rho_j\, \boldsymbol{e}_i = -\tfrac{1}{2} \in_{ijk} \rho_j\, \omega_k\, \boldsymbol{e}_i = -\tfrac{1}{2} \boldsymbol{\rho} \times \boldsymbol{\omega} = \tfrac{1}{2} \boldsymbol{\omega} \times \boldsymbol{\rho} .$$

Moreover, we get

$$\frac{d}{dt}|\boldsymbol{\rho}|^2 = \frac{d}{dt}(\boldsymbol{\rho},\boldsymbol{\rho}) = 2\left(\boldsymbol{\rho},\frac{d}{dt}\boldsymbol{\rho}\right) = 2(\boldsymbol{\rho},\boldsymbol{R}\boldsymbol{\rho}) = \boldsymbol{\rho}\cdot(\boldsymbol{\omega}\times\boldsymbol{\rho}) = \boldsymbol{0}$$

i.e., the distance between two (arbitrary) fluid elements is fixed and there is no shape deformation.

The skew-symmetric part of the velocity gradient describes pure rigid rotation of the fluid and the local angular velocity is equal $\frac{1}{2}\boldsymbol{\omega}$.

Deformation of fluid elements

The rate of change of the relative position vector (or – equivalently – the velocity of the relative motion of two infinitely close fluid elements) can be expressed by the formula

$$\frac{d}{dt}\boldsymbol{\rho} = \underbrace{\boldsymbol{D}\boldsymbol{\rho}}_{\text{deformation}} + \underbrace{\boldsymbol{R}\boldsymbol{\rho}}_{\text{rigid rotation}} = \boldsymbol{D}\boldsymbol{\rho} + \tfrac{1}{2}\boldsymbol{\omega}\times\boldsymbol{\rho}.$$

The first terms consists the symmetric tensor $\boldsymbol{D}$, called the **deformation rate tensor**.

The tensor $\boldsymbol{D}$ can be expressed as the sum of the **spherical part** $\boldsymbol{D}_{SPH}$ and the **deviatoric part** $\boldsymbol{D}_{DEV}$

$$\boldsymbol{D} = \boldsymbol{D}_{SPH} + \boldsymbol{D}_{DEV}$$

The **spherical part** $\boldsymbol{D}_{SPH}$ **describes pure volumetric deformation** (uniform expansion or contraction without any shape changes) and it defined as

$$\boldsymbol{D}_{SPH} = \frac{1}{3}\underbrace{tr\boldsymbol{D}}_{\text{trace of } \boldsymbol{D}}\cdot\boldsymbol{I} = \frac{1}{3}(\nabla\cdot\boldsymbol{v})\boldsymbol{I} \quad \Rightarrow \quad (\boldsymbol{D}_{SPH})_{ij} = \frac{1}{3}\frac{\partial v_k}{\partial x_k}\delta_{ij},$$

Note that

$$tr\boldsymbol{D}_{SPH} = \tfrac{1}{3}(\nabla\cdot\boldsymbol{v})\cdot\underbrace{tr\boldsymbol{I}}_{3} = \nabla\cdot\boldsymbol{v}.$$

The second part $\boldsymbol{D}_{DEV}$ describes **shape changes which preserve the volume**.

We have

$$\boldsymbol{D}_{DEV} = \boldsymbol{D} - \frac{1}{3} div\, \boldsymbol{\upsilon} \cdot \boldsymbol{I} \quad \Rightarrow \quad (\boldsymbol{D}_{DEV})_{ij} = \frac{1}{2}\left(\frac{\partial \upsilon_i}{\partial x_j} + \frac{\partial \upsilon_j}{\partial x_i}\right) - \frac{1}{3}\frac{\partial \upsilon_k}{\partial x_k}\delta_{ij}$$

and

$$tr\,\boldsymbol{D}_{DEV} = tr\,\boldsymbol{D} - tr\,\boldsymbol{D}_{SPH} = 0$$

To explain the **geometric interpretation** of both parts of the deformation rate tensor, consider the deformation of a small, initially rectangular portion of a fluid in two dimensions. Assume there is no rotation part and thus we can write

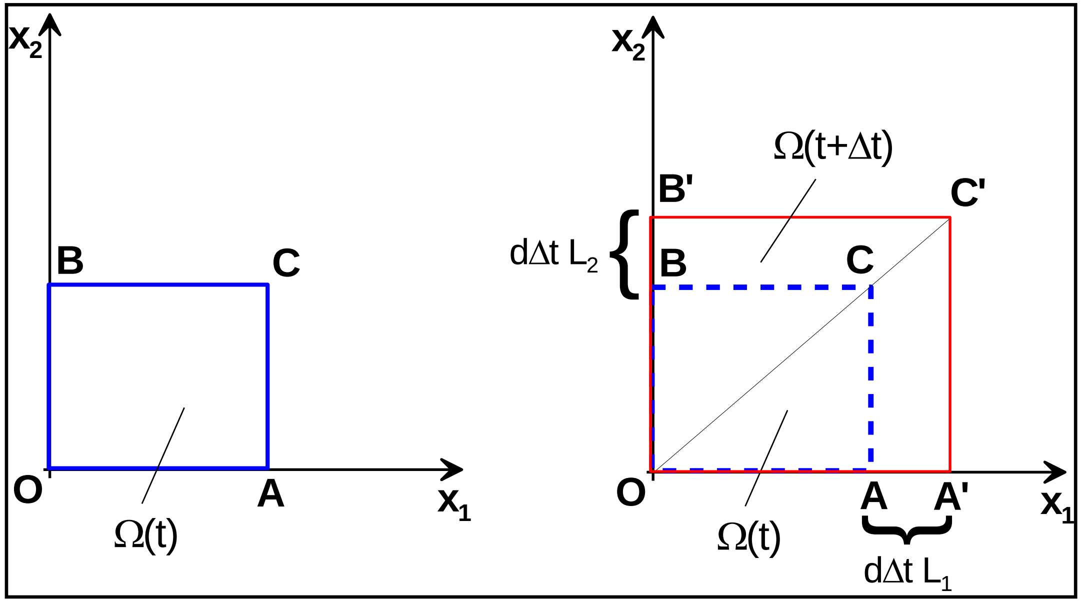
$$\tfrac{d}{dt}\boldsymbol{\rho} = \boldsymbol{D}\boldsymbol{\rho} = \boldsymbol{D}_{SPH}\boldsymbol{\rho} + \boldsymbol{D}_{DEV}\boldsymbol{\rho} \; .$$

For a short time interval Δt the above relation yields

$$\boldsymbol{\rho}(t+\Delta t) = \boldsymbol{\rho}(t) + \underbrace{\boldsymbol{D}_{SPH}\boldsymbol{\rho}\,\Delta t}_{\Delta\boldsymbol{\rho}_1} + \underbrace{\boldsymbol{D}_{DEV}\boldsymbol{\rho}\,\Delta t}_{\Delta\boldsymbol{\rho}_2} + O(\Delta t^2)$$

Consider the 2D case when only volumetric part of the deformation exists (see picture).

We have $\boldsymbol{D}_{SPH} = \begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix}, \quad tr\,\boldsymbol{D} = 2\,d$



The relative position vector at the time instant $t+\Delta t$ is expressed as

$$\boldsymbol{\rho}(t+\Delta t) = (1 + d\,\Delta t)\,\boldsymbol{\rho}(t) + O(\Delta t^2)$$

The shape of the volume is preserved because the above formula describes the isotropic expansion/contraction. **The volume of the region** $Vol_\Omega(t) = L_1 L_2$ **has been changed to**

$$Vol_\Omega(t+\Delta t) = L_1 L_2 (1 + d\,\Delta t)^{\,2} = Vol_\Omega(t)(1 + 2d\,\Delta t) + O(\Delta t^2),$$

and $$\frac{1}{Vol_\Omega(t)} \lim_{\Delta t \to 0} \frac{Vol_\Omega(t+\Delta t) - Vol_\Omega(t)}{\Delta t} = 2\,d = tr\,\boldsymbol{D} = \nabla \cdot \boldsymbol{v}$$

Assume now that the spherical part of the deformation rate tensor is absent. The deviatoric part of this tensor in a 2D flow can be written as follows

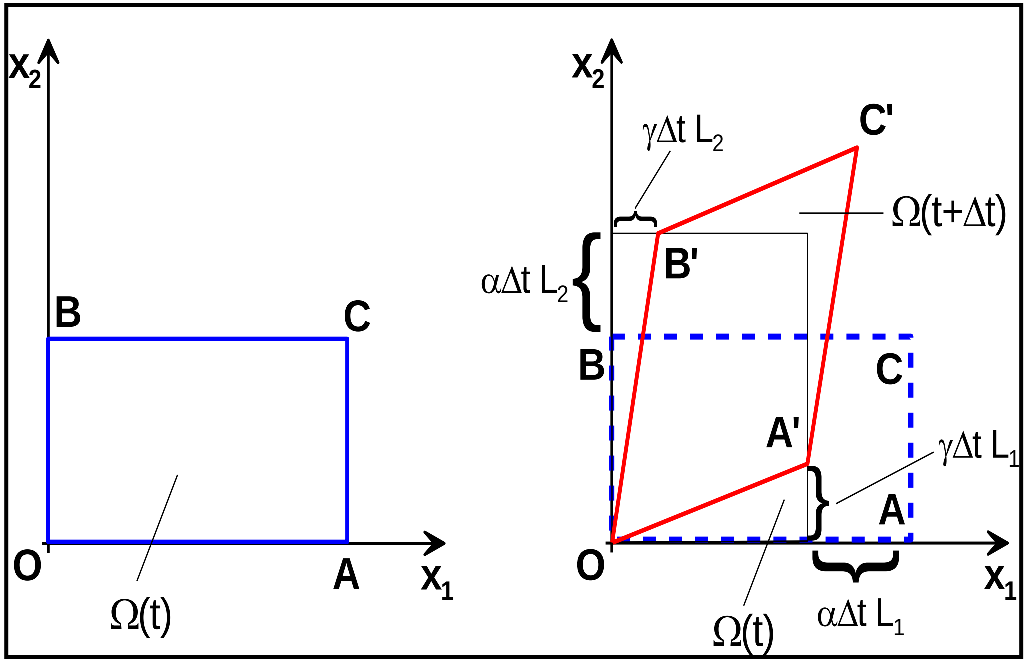
$$\boldsymbol{D}_{DEV}=\begin{bmatrix} d_{11}-\frac{1}{2}d & d_{12} \\ d_{12} & d_{22}-\frac{1}{2}d \end{bmatrix}=\begin{bmatrix} \frac{1}{2}(d_{11}-d_{22}) & d_{12} \\ d_{12} & \frac{1}{2}(d_{22}-d_{11}) \end{bmatrix}=\begin{bmatrix} -\alpha & \gamma \\ \gamma & \alpha \end{bmatrix}.$$

The fluid deformation during the short time interval can be now expressed as

$$\boldsymbol{\rho}(t+\Delta t)=(\boldsymbol{I}+\Delta t\,\boldsymbol{D}_{DEV})\,\boldsymbol{\rho}(t)+O(\Delta t^2)$$

or in the explicit form as

$$\begin{cases} x_1(t+\Delta t)=(1-\alpha\,\Delta t)\,x_1(t)+\gamma\,\Delta t\,x_2(t) \\ x_2(t+\Delta t)=\gamma\,\Delta t\,x_1(t)+(1+\alpha\,\Delta t)\,x_2(t) \end{cases}$$



Note the presence of **shear**, which manifests in the **change of the angles** between the position vectors corresponding to different fluid elements in the deforming region.

Let's compute again the change of the **volume of the fluid region** during such deformation.

We get

$$Vol_{\Omega}(t+\Delta t)=\begin{vmatrix}1-\alpha\,\Delta t & \gamma\,\Delta t\\ \gamma\,\Delta t & 1+\alpha\,\Delta t\end{vmatrix}L_1L_2=L_1L_2(1-\alpha^2\Delta t^2-\gamma^2\Delta t^2)=,$$

$$=Vol_{\Omega}(t)+O(\Delta t^2)$$

so

$$\frac{1}{Vol_{\Omega}(t)}\lim_{\Delta t\to 0}\frac{Vol_{\Omega}(t+\Delta t)-Vol_{\Omega}(t)}{\Delta t}=0.$$

We conclude that this time **the instantaneous rate of the volume change is zero**. Thus, instantaneously, the **deviatoric part of the deformation describes pure shear** (no expansion/contraction).

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