LECTURE 4

LINEAR MOMENTUM PRINCIPLE AND GENERAL EQUATION OF MOTION.
THE ANGULAR MOMENTUM PRINCIPLE.

DERIVATION OF EQUATION OF FLUID MOTION

We have already mentioned that the Linear Momentum Principle is formulated by setting $h = v$ in the general conservation law introduced in the Lecture 3. Since the "sources" for linear momentum are the acting external forces, we get the following equality

$$\frac{d}{dt} \int_{\Omega} \rho v \, dV - \left(- \int_{\partial\Omega} \rho v (v \cdot n) \, dS \right) = F_S + F_V$$

As before, we can change order of time differentiation and integration. Then, after insertion of the expressions for the forces, we get

$$\int_{\Omega} \frac{\partial}{\partial t} (\rho v) \, dV + \int_{\partial\Omega} \rho v (v \cdot n) \, dS = \int_{\partial\Omega} \sigma \, dS + \int_{\Omega} \rho f \, dV$$

Later we will show that the surface stress can be expressed as the following matrix-vector product

$$\sigma = \Xi n$$

More precisely: the stress vector σ is obtained by the application of the stress tensor Ξ to the normal vector n. **We will discuss this problems in details in the Lecture 6.**

It the case of idealized fluid with no viscosity (the Euler fluid) the stress vector has always only normal component directed into the volume and expressed by the local value of pressure by the same formula as in the statics, i.e.

$$\boldsymbol{\sigma} = -p\boldsymbol{n}$$

It means that the stress tensor is particularly simple, namely $\boldsymbol{\Xi} = -p\boldsymbol{I}$, where $\boldsymbol{I}$ stands for the unary matrix (identity tensor).

Our next goal is to derive a general differential equation of motion. The procedure as usual consists in transformation all surface integrals to volume ones. Then, thanks to arbitrariness of the volume Ω, we conclude that the integrand vanishes everywhere in the flow domain.

The appropriate calculations are more laborious that in the case of mass conservation. We will start with transforming the surface integral describing the flux of the linear momentum through the boundary:

$$\int_{\partial\Omega} (\rho\boldsymbol{v})(\boldsymbol{v}\cdot\boldsymbol{n})\,dS = \boldsymbol{e}_i \int_{\partial\Omega} \rho\upsilon_i\upsilon_j n_j\,dS \underset{\substack{GGO \\ Theor.}}{=} \boldsymbol{e}_i \int_{\Omega} \frac{\partial}{\partial x_j}(\underbrace{\rho\upsilon_i\upsilon_j}_{(\rho\boldsymbol{v}\otimes\boldsymbol{v})_{ij}})\,dV = \int_{\Omega} \nabla\cdot(\rho\boldsymbol{v}\otimes\boldsymbol{v})\,dV$$

Note that we have obtained a volume integral from the vector field which is a **tensor divergence** from the **momentum flux tensor** $\boldsymbol{\Pi} = \rho\boldsymbol{v}\otimes\boldsymbol{v}$, where the operator $\otimes$ denotes the tensor (or dyadic) product of two vectors.

Next let us consider the surface integral representing the surface force. The transformation to the volume integral is conducted as follows

$$\int_{\partial\Omega} \boldsymbol{\sigma}\, dS = \int_{\partial\Omega} \boldsymbol{\varXi}\boldsymbol{n}\, dS = \boldsymbol{e}_i \int_{\partial\Omega} \varXi_{ij} n_j\, dS \underset{\substack{GGO \\ Theor.}}{=} \boldsymbol{e}_i \int_{\Omega} \frac{\partial}{\partial x_j} \varXi_{ij}\, dV \equiv \int_{\Omega} \nabla\cdot\boldsymbol{\varXi}\, dV$$

Note that the volumetric integration is applied to the vector which is the **tensor divergence** of the **stress tensor field** $\boldsymbol{\varXi}$. In particular case of the ideal (inviscid) fluid, we get

$$\int_{\partial\Omega} \boldsymbol{\sigma}\, dS = \int_{\partial\Omega} (-p\boldsymbol{I})\boldsymbol{n}\, dS = -\boldsymbol{e}_i \int_{\partial\Omega} p\, \delta_{ij} n_j\, dS \underset{\substack{GGO \\ Theor.}}{=} -\boldsymbol{e}_i \int_{\Omega} \underbrace{\delta_{ij} \frac{\partial}{\partial x_j} p}_{\frac{\partial}{\partial x_i} p}\, dV \equiv -\int_{\Omega} \nabla p\, dV$$

Insertion of the volumetric integrals into the original equality yields in the general case

$$\int_{\Omega} \left\{ \frac{\partial}{\partial t}(\rho\boldsymbol{v}) + \nabla\cdot(\rho\boldsymbol{v}\otimes\boldsymbol{v} - \boldsymbol{\varXi}) - \rho\boldsymbol{f} \right\} dV = \boldsymbol{0}$$

Thus, we conclude that the following differential equation must be satisfied at each point in fluid domain

$$\frac{\partial}{\partial t}(\rho \boldsymbol{v}) + \nabla \cdot (\rho \boldsymbol{v} \otimes \boldsymbol{v} - \boldsymbol{\Xi}) = \rho \boldsymbol{f}$$

We have obtained the **general equation of motion in the conservative form.**

For the ideal fluid $\boldsymbol{\Xi} = -p\boldsymbol{I}$, hence the equation of motion reads

$$\frac{\partial}{\partial t}(\rho \boldsymbol{v}) + \nabla \cdot (\rho \boldsymbol{v} \otimes \boldsymbol{v} + p\boldsymbol{I}) = \rho \boldsymbol{f}$$

The conservative form of the equation of motion is not the only one possible. There exists a simpler form. It is worth noting that this simpler form can be obtained directly from an alternative method of derivation, which is based on the usage of material rather than control volumes. Here, to obtain simplified form of the equation of motion we perform the following calculations

$$\frac{\partial}{\partial t}(\rho \boldsymbol{v}) + \nabla \cdot (\rho \boldsymbol{v} \otimes \boldsymbol{v}) = \boldsymbol{e}_i \left[\frac{\partial}{\partial t}(\rho \upsilon_i) + \frac{\partial}{\partial x_j}(\rho \upsilon_i \upsilon_j) \right] =$$

$$= \boldsymbol{e}_i \left[\upsilon_i \frac{\partial}{\partial t}\rho + \rho \frac{\partial}{\partial t}\upsilon_i + \upsilon_i \frac{\partial}{\partial x_j}(\rho \upsilon_j) + \rho \upsilon_j \frac{\partial}{\partial x_j}\upsilon_i \right] =$$

$$= \boldsymbol{v} \cdot [\underbrace{\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho \boldsymbol{v})}_{0!!!(mass\ conservation)}] + \rho[\frac{\partial}{\partial t}\boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}] = \rho \boldsymbol{a}$$

Note that the expression in the second square bracket is nothing else but the fluid **acceleration**

$$a = \tfrac{\partial}{\partial t} v + (v \cdot \nabla) v$$

The general equation of motion can be now written as

$$\boxed{\rho [\tfrac{\partial}{\partial t} v + (v \cdot \nabla) v] = \nabla \cdot \Xi + \rho f}$$

For the **ideal (inviscid) fluid** the general equation reduces to the **Euler Equation**

$$\boxed{\rho [\tfrac{\partial}{\partial t} v + (v \cdot \nabla) v] = -\nabla p + \rho f}$$

which is often delivered in the formed divided by density ρ

$$\tfrac{\partial}{\partial t} v + (v \cdot \nabla) v = -\tfrac{1}{\rho} \nabla p + f$$

Later, we will consider also consider the case of a viscous fluid.

ANGULAR MOMENTUM PRINCIPLE. SYMMETRY OF STRESS TENSOR.

The angular momentum of the fluid contained in the volume Ω (computed with respect to the origin of the reference frame) is equal

$$\boldsymbol{K} = \int_{\Omega} \boldsymbol{x} \times \rho \boldsymbol{v} \, dV$$

Following the general rule, the net rate of time variation of $\boldsymbol{K}$ is expressed by the formula

$$\left.\frac{d\boldsymbol{K}}{dt}\right|_{production} = \frac{d}{dt} \int_{\Omega} \boldsymbol{x} \times \rho \boldsymbol{v} \, dV + \int_{\partial\Omega} (\boldsymbol{x} \times \rho \boldsymbol{v})(\boldsymbol{v} \cdot \boldsymbol{n}) \, dS$$

The Angular Momentum Principle (AMP)

$$\left.\frac{d\boldsymbol{K}}{dt}\right|_{production} = \boldsymbol{M}_S + \boldsymbol{M}_V$$

takes the following form

$$\frac{d}{dt} \int_{\Omega} \boldsymbol{x} \times \rho \boldsymbol{v} \, dV + \int_{\partial\Omega} (\boldsymbol{x} \times \rho \boldsymbol{v})(\boldsymbol{v} \cdot \boldsymbol{n}) \, dS = \int_{\partial\Omega} \boldsymbol{x} \times \boldsymbol{\Xi}\boldsymbol{n} \, dS + \int_{\Omega} \boldsymbol{x} \times \rho \boldsymbol{f} \, dV$$

As usual, we would like to derive a differential form of this principle. Again, the time differentiation operation can be done under the sign of integral, i.e.

$$\frac{d}{dt}\int_{\Omega} \boldsymbol{x} \times \rho\boldsymbol{v}\, dV = \int_{\Omega} \boldsymbol{x} \times \frac{\partial}{\partial t}(\rho\boldsymbol{v})\, dV$$

As usual, we need to transform surface integral into the volume integrals. Note that the integral in the left–hand side of the above equation can be write as

$$\int_{\partial\Omega} (\boldsymbol{x} \times \rho\boldsymbol{v})(\boldsymbol{v} \cdot \boldsymbol{n})\, dS = \int_{\partial\Omega} \boldsymbol{x} \times [(\underbrace{\rho\boldsymbol{v} \otimes \boldsymbol{v}}_{tensor})\boldsymbol{n}]\, dS$$

Thus, both surface integrals in the AMP are particular cases of

$$\boldsymbol{j} = \int_{\partial\Omega} \boldsymbol{x} \times \boldsymbol{A}\boldsymbol{n}\, dS$$

Transformation of such integral to the volume one is a good opportunity to practice the index formalism.

$$\boldsymbol{j} = j_i \boldsymbol{e}_i \equiv \boldsymbol{e}_i \int_{\partial\Omega} (\boldsymbol{x} \times \boldsymbol{A}\boldsymbol{n})_i \, dS = \boldsymbol{e}_i \int_{\partial\Omega} \in_{ijk} x_j \underbrace{(a_{km} n_m)}_{(\boldsymbol{A}\boldsymbol{n})_k} dS \underset{GGO\,Th.}{=} \boldsymbol{e}_i \int_{\Omega} \in_{ijk} \frac{\partial}{\partial x_m}(x_j a_{km}) \, dV =$$

$$= \boldsymbol{e}_i \int_{\Omega} \in_{ijk} \underset{\delta_{jm}}{\frac{\partial x_j}{\partial x_m}} a_{km} dV + \boldsymbol{e}_i \int_{\Omega} \in_{ijk} x_j \underset{(\nabla\cdot\boldsymbol{A})_k}{\frac{\partial a_{km}}{\partial x_m}} dV = \boldsymbol{e}_i \int_{\Omega} \in_{ijk} a_{kj} dV + \left(\int_{\Omega} \boldsymbol{x} \times \nabla\cdot\boldsymbol{A} \, dV \right)_i \boldsymbol{e}_i =$$

$$= \boldsymbol{e}_1 \int_{\Omega} (a_{32} - a_{23}) dV + \boldsymbol{e}_2 \int_{\Omega} (a_{13} - a_{31}) dV + \boldsymbol{e}_3 \int_{\Omega} (a_{21} - a_{12}) dV + \int_{\Omega} \boldsymbol{x} \times \nabla\cdot\boldsymbol{A} \, dV$$

Note that if the matrix $\boldsymbol{A}$ is symmetric ($\boldsymbol{A} = \boldsymbol{A}^T$) then three first terms are zero. This is the case for the first of our integrals, where

$$\boldsymbol{A} = \rho \boldsymbol{v} \otimes \boldsymbol{v} \quad \Rightarrow \quad a_{ij} = \rho v_i v_j$$

and hence $a_{ij} = a_{ji}$, $i, j = 1,2,3$. Thus

$$\int_{\partial\Omega} \boldsymbol{x} \times (\rho \boldsymbol{v} \otimes \boldsymbol{v}) \boldsymbol{n} \, dS = \int_{\Omega} \boldsymbol{x} \times \nabla\cdot(\rho \boldsymbol{v} \otimes \boldsymbol{v}) \, dV$$

In the second surface integral we have $A = \Xi$ and

$$\int_{\partial\Omega} x \times \Xi n \, dS = \int_{\Omega} x \times \nabla \cdot \Xi \, dV + e_1 \int_{\Omega} (\Xi_{32} - \Xi_{23}) dV + e_2 \int_{\Omega} (\Xi_{13} - \Xi_{31}) dV +$$

$$+ e_3 \int_{\Omega} (\Xi_{21} - \Xi_{12}) dV$$

Insertion to the AMP integral formula brings the equality

$$\int_{\Omega} x \times [\tfrac{\partial}{\partial t}(\rho v) + \nabla \cdot (\rho v \otimes v - \Xi) - \rho f] dV + e_1 \int_{\Omega} (\Xi_{32} - \Xi_{23}) dV +$$

$$+ e_2 \int_{\Omega} (\Xi_{13} - \Xi_{31}) dV + e_3 \int_{\Omega} (\Xi_{21} - \Xi_{12}) dV = 0$$

But $\tfrac{\partial}{\partial t}(\rho v) + \nabla \cdot (\rho v \otimes v - \Xi) - \rho f = 0$ (eq. of motion)! Since Ω is arbitrary, we get

$$\Xi_{32} = \Xi_{23} \quad , \quad \Xi_{12} = \Xi_{21} \quad , \quad \Xi_{13} = \Xi_{31}$$

i.e., at any point in the flow domain that the stress tensor Ξ is symmetric, i.e., $\Xi = \Xi^T$!

CONCLUSION …

If the stress tensor is symmetric then the validity of the Angular Momentum Principle follows from the Linear Momentum Principle. In other words: **in the fluid model where symmetry of the stress tensor is guaranteed, only Linear Momentum Principle is necessary to describe the fluid motion.**