PROGRAM ROZWOJOWY
POLITECHNIKI WARSZAWSKIEJ

LECTURE 2
KINEMATICS OF FLUID – PART 1

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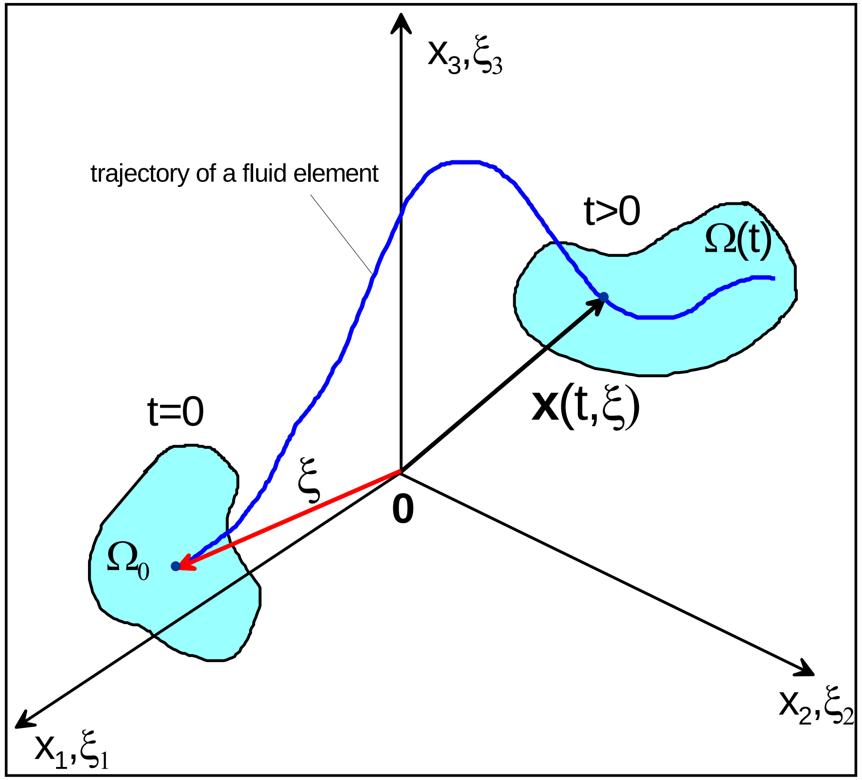
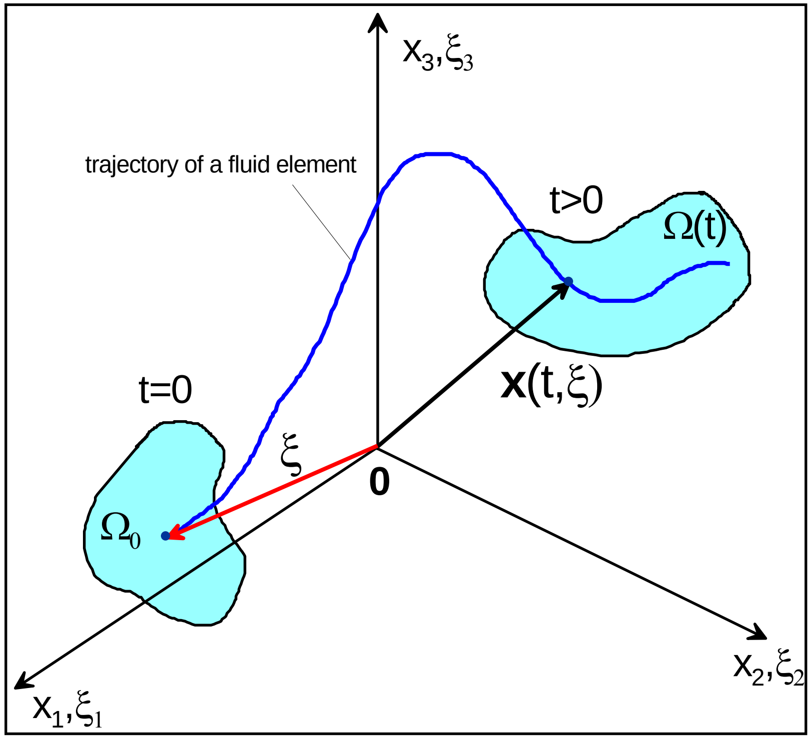
Lagrange and Euler views on the fluid motion. Fluid velocity and acceleration

Fluid element is defined as an individual and infinitely small portion of the fluid. Each fluid element is characterized by its instantaneous location (or position) vector $\boldsymbol{x}$, which is the function of time t and the initial position $\boldsymbol{\xi}$ of the element, i.e. its location at the time instant t = 0. Thus we have $\boldsymbol{x} = \boldsymbol{x}(t, \boldsymbol{\xi})$ and in particular $\boldsymbol{\xi} = \boldsymbol{x}(0, \boldsymbol{\xi})$.



If we fix $\boldsymbol{\xi} = \boldsymbol{\xi}_*$ then the function $\boldsymbol{x} = \boldsymbol{x}(t, \boldsymbol{\xi}_*)$ describes some line in E^3 called the **trajectory of the fluid element.**

For the fixed time $t \geq 0$ the function $\boldsymbol{x} = \boldsymbol{x}(t, \xi)$ describes the transformation of the region filled with the fluid at the time $t = 0$ – let's denote it Ω_0 - to the region $\Omega(t) = \boldsymbol{x}(t, \Omega_0)$ **containing the same fluid at some later time** t_*. Thus, the region $\Omega(t)$ is the image of Ω_0 with respect to the transformation $\boldsymbol{x} = \boldsymbol{x}(t, \xi)$; we call $\Omega(t)$ the **material volume** because all the time it consists of the **same fluid elements**, i.e. those which belong originally to Ω_0.



Note: two, originally different, fluid elements cannot drop into the same point where the velocity is not zero, i.e., only one trajectory can go through such point. These condition can be described mathematically as follows. If $\boldsymbol{x}_1 = \boldsymbol{x}(t, \xi)$ and $\boldsymbol{x}_2 = \boldsymbol{x}(t + \tau, \xi)$ then the following **group property** holds

$$\boldsymbol{x}_2 = \boldsymbol{x}(t + \tau, \xi) = \boldsymbol{x}[\tau, \boldsymbol{x}(t, \xi)] = \boldsymbol{x}(\tau, \boldsymbol{x}_1)$$

Any fluid motion can be described using either Lagrangian or Eulerian viewpoint.

Lagrange viewpoint: each fluid element is identified uniquely by its position at $t = 0$, i.e. by the vector $\boldsymbol{\xi}$. All kinematical and dynamic quantities are described as functions of time and the Lagrangian coordinates ξ_1, ξ_2 and ξ_3.

The velocity of the fluid element is defined as

$$\boldsymbol{V}(t,\boldsymbol{\xi}) := \lim_{\Delta t \to 0} \frac{\boldsymbol{x}(t+\Delta t,\boldsymbol{\xi}) - \boldsymbol{x}(t,\boldsymbol{\xi})}{\Delta t} \equiv \frac{\partial \boldsymbol{x}}{\partial t}(t,\boldsymbol{\xi}) \qquad (\boldsymbol{\xi} - \text{fixed})$$

Fluid acceleration is defined as

$$\boldsymbol{a}(t,\boldsymbol{\xi}) := \lim_{\Delta t \to 0} \frac{\boldsymbol{V}(t+\Delta t,\boldsymbol{\xi}) - \boldsymbol{V}(t,\boldsymbol{\xi})}{\Delta t} \equiv \frac{\partial \boldsymbol{V}}{\partial t}(t,\boldsymbol{\xi}) = \frac{\partial^2 \boldsymbol{x}}{\partial t^2}(t,\boldsymbol{\xi})$$

Euler viewpoint: the velocity, acceleration and other kinematical or dynamical quantities are described as functions of **time** t and the **position of the fluid element** at this time instant (not at the initial time!), i.e. by the coordinates x_1, x_2 and $\boldsymbol{x_3}$ of the vector $\boldsymbol{x}$.

The **velocity field** is the function of time and space coordinates $\boldsymbol{v} = \boldsymbol{v}(t, \boldsymbol{x})$.

The relations between two different viewpoints can be written as

Euler to Lagrange: $\boldsymbol{V}(t, \boldsymbol{\xi}) = \boldsymbol{v}[t, \boldsymbol{x}(t, \boldsymbol{\xi})]$

Lagrange to Euler: $\boldsymbol{v}(t, \boldsymbol{x}) = \boldsymbol{V}[t, \underbrace{\boldsymbol{\xi}(t, \boldsymbol{x})}_{\text{inverse transform}}]$

The Euler form of the fluid acceleration will be considered later.

TRAJECTORIES OF FLUID ELEMENTS

Lagrange: $\frac{\partial}{\partial t}\boldsymbol{x}(t,\boldsymbol{\xi}) = \boldsymbol{V}(t,\boldsymbol{\xi})$ ($\boldsymbol{\xi}$ – fixed parameter).

Thus $$\boldsymbol{x}(t,\boldsymbol{\xi}) = \boldsymbol{\xi} + \int_0^t \boldsymbol{V}(\tau,\boldsymbol{\xi})\,d\tau .$$

We have obtained direct integral formula which can be calculated numerically (e.g. using the trapezoidal integration rule)

Euler: we have the following initial value problem

$$\begin{cases} \frac{d}{dt}\boldsymbol{x} = \boldsymbol{\upsilon}[t, x(t)] \\ \boldsymbol{x}(0) = \boldsymbol{\xi} \end{cases} \Rightarrow \begin{cases} \frac{d}{dt}x_j = \upsilon_j(t, x_1, x_2, x_3)\ ,\ \ j = 1,2,3. \\ x_j(0) = \xi_j \end{cases}$$

Typically, the above Initial Value Problem has to be solved numerically (e.g. using the Runge-Kutta methods)

STREAMLINES OF THE VELOCITY FIELD

The streamline: line *l* such that for every point P on *l* the velocity vector at the point P is tangent to *l*.

The **tangency condition** can be written as

$$\frac{dx_1}{\upsilon_1(t,x_1,x_2,x_3)} = \frac{dx_2}{\upsilon_2(t,x_1,x_2,x_3)} = \frac{dx_3}{\upsilon_3(t,x_1,x_2,x_3)}$$

The above equalities can be view as the differential equivalent of the “edge” description of the 2-parameter family of lines in 3-dimensional space, namely

$$\begin{cases} \mathfrak{F}_1(t,x_1,x_2,x_3,C_1,C_2)=0 \\ \mathfrak{F}_2(t,x_1,x_2,x_3,C_1,C_2)=0 \end{cases}$$

In the above, time t is treated as the fixed parameter. **In other words, the pattern of streamlines is determined for each time instant separately and – in general – the form of the streamlines at different time instants is not the same.**

The practical method of computing the streamlines is to "freeze" time and "inject" the marker particles into the "frozen" velocity field. The movement of the marker particle injected in the point $\boldsymbol{x}_0$ is described by the following initial value problem

$$\begin{cases} \frac{d}{d\tau}\boldsymbol{x}(\tau) = \boldsymbol{v}[t, \boldsymbol{x}(\tau)] \\ \boldsymbol{x}(\tau = 0) = \boldsymbol{x}_0 \end{cases}$$

where physical time t is fixed ("frozen") and the variable τ is the "pseudo-time". In other words, **the streamlines are the trajectories of the marker particles moving in the frozen velocity field**.

We conclude immediately that **fluid element trajectories and the streamlines are identical if the velocity field does not depend explicitly on time, i.e. if the flow is stationary**.

EXAMPLES

(1) Stationary two-dimensional flow $\boldsymbol{v}(x_1, x_2) = -x_2\boldsymbol{e}_1 + x_1\boldsymbol{e}_2 \equiv [-x_2, x_1]$

Streamlines: $\dfrac{dx_1}{-x_2} = \dfrac{dx_2}{x_1} \Rightarrow \quad x_1\,dx_1 + x_2\,dx_2 = 0 \Rightarrow x_1^2 + x_2^2 = R^2\ ,\ R \geq 0.$

We have obtained the family of the **concentric circles**.

Trajectories: $\begin{cases} \frac{d}{dt}x_1 = -x_2\ ,\ \frac{d}{dt}x_2 = x_1 \\ x_1(0) = R\ ,\ x_2(0) = 0 \end{cases}$

The solution is

$$x_1(t) = R\cos(t)\ ,\ x_2(t) = R\sin(t)$$

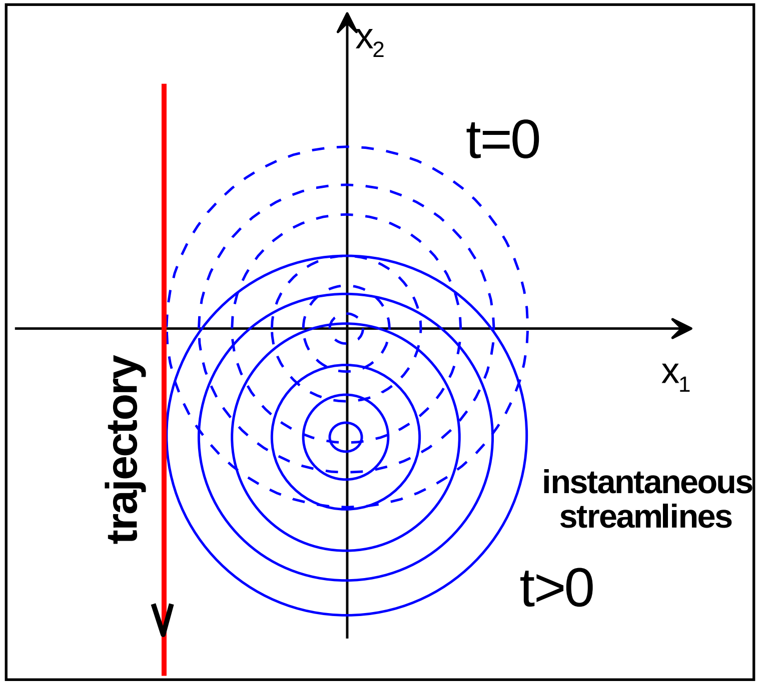
which is the parametric form of the circle $x_1^2 + x_2^2 = R^2\ ,\ R \geq 0$.

(2) Nonstationary (or unsteady) flow $\boldsymbol{v}(t,x_1,x_2)=(-x_2-t)\boldsymbol{e}_1+x_1\boldsymbol{e}_2\equiv[-x_2-t,x_1]$

Streamlines:

$$\frac{dx_1}{-x_2-t}=\frac{dx_2}{x_1}\ \Rightarrow\ x_1\,dx_1+(x_2+t)\,dx_2=0\ \Rightarrow\ x_1^2+(x_2+t)^2=C+t^2\equiv R^2\ ,\ C\geq -t^2.$$

Again: the family of concentric circles but **the pattern of the streamlines moves down along the 0x_2 axis with the steady speed equal –1** (see figure).



Trajectories:

$$\begin{cases}\frac{d}{dt}x_1=-x_2-t\ ,\ \frac{d}{dt}x_2=x_1\\ x_1(0)=x_{10}\ ,\ x_2(0)=x_{20}\end{cases}$$

The solution

$$\begin{cases}x_1(t)=(x_{10}+1)\cos(t)-x_{20}\sin(t)-1\\ x_2(t)=(x_{10}+1)\sin(t)+x_{20}\cos(t)-t\end{cases}$$

Consider $x_{10}=-1$ and $x_{20}=0$. Then $x_1(t)=-1$ and $x_2(t)=-t$ so the fluid element moves down the straight vertical line $x_1=-1$ with the steady velocity equal to -1.

The trajectories in the unsteady flow can be quite different that the streamlines!

SUBSTANTIAL (MATERIAL, LAGRANGE, LIE) DERIVATIVE

Consider a sufficiently regular scalar field $f = f(t, \boldsymbol{x}) = f(t, x_1, x_2, x_3)$. For an observer moving with a given fluid element the value of this field is a time dependent quantity described by the composite function

$$F(t) := f[t, \boldsymbol{x}(t, \boldsymbol{\xi})]$$

The rate of change in time of the field f seen by such observer moving with the fluid is called **the substantial (material, Lagrange, Lie or full) derivative of the field f.**

Mathematically, we have

$$\frac{Df}{Dt}[t, \boldsymbol{x}(t,\xi)] := \frac{dF}{dt}(t) = \frac{\partial f}{\partial t}[t, \boldsymbol{x}(t,\xi)] + \frac{\partial f}{\partial x_1}[t, \boldsymbol{x}(t,\xi)]\frac{\partial x_1}{\partial t}(t,\xi) +$$

$$+\frac{\partial f}{\partial x_2}[t, \boldsymbol{x}(t,\xi)]\frac{\partial x_2}{\partial t}(t,\xi) + \frac{\partial f}{\partial x_3}[t, \boldsymbol{x}(t,\xi)]\frac{\partial x_3}{\partial t}(t,\xi) = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_1}\upsilon_1 + \frac{\partial f}{\partial x_2}\upsilon_2 + \frac{\partial f}{\partial x_3}\upsilon_3\right)[t, \boldsymbol{x}(t,\xi)]$$

where we have used the relation $\quad \dfrac{\partial x_j}{\partial t}(t,\boldsymbol{\xi}) = V_j(t,\boldsymbol{\xi}) = \upsilon_j[t, \boldsymbol{x}(t,\boldsymbol{\xi})] \;,\; j = 1,2,3.$

Since the arguments at both sides are the same, we have obtained the scalar field which can be written using "nabla" operator as

$$\frac{Df}{Dt} = \underset{\substack{\text{local}\\\text{derivative}}}{\frac{\partial f}{\partial t}} + \underbrace{\boldsymbol{\upsilon} \cdot \nabla f}_{\substack{\text{convective}\\\text{derivative}}},$$

or in the index notation (summation convention is assumed)

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \upsilon_j \frac{\partial f}{\partial x_j}$$

- The **first term** in the right-hand side of the above definition is called a **local derivative**. It "measures" the rate of change of the field f due to its **explicit time dependence** at a fixed space location. It $f = f(\mathbf{x}) = f(x_1, x_2, x_3)$ we say that f is **stationary** (or steady) and the local derivative $\partial f / \partial t$ **vanishes identically**.
- The **second term** is called the **convective derivative** of the field f. **It is generally nonzero even if the field f is stationary**. It measures the rate of change due to the **movement of the observer**. This part of the substantial derivative vanishes identically if the field f is uniform in space, i.e. its instantaneous value at each point is the same.

ACCELERATION – EULER VIEW

Consider the acceleration of fluid elements in **Euler** description. In order to calculate the acceleration **we need to differentiate the velocity along the trajectories of fluid elements**. We have

$$\boldsymbol{a}(t,\boldsymbol{x}) = \frac{D\boldsymbol{\upsilon}}{Dt} = \frac{D\upsilon_i}{Dt}\boldsymbol{e}_i = \left(\frac{\partial \upsilon_i}{\partial t} + \upsilon_j \frac{\partial \upsilon_i}{\partial x_j}\right)\boldsymbol{e}_i = \frac{\partial \boldsymbol{\upsilon}}{\partial t} + \upsilon_j \frac{\partial \boldsymbol{\upsilon}}{\partial x_j}$$

We see that the **acceleration is the vector field**. In popular notation, the **convective part of the acceleration** is written using the **formal inner product** of the velocity field and nabla operator

$$\boldsymbol{a}(t,\boldsymbol{x}) = \frac{D\boldsymbol{\upsilon}}{Dt} = \frac{\partial \boldsymbol{\upsilon}}{\partial t} + (\boldsymbol{\upsilon}\cdot\nabla)\boldsymbol{\upsilon}.$$

An alternative way of expressing the fluid acceleration is the Lamb-Gromeko form

$$\boldsymbol{a}(t,\boldsymbol{x}) = \frac{\partial \boldsymbol{\upsilon}}{\partial t} + \nabla\left(\tfrac{1}{2}\upsilon^2\right) + \boldsymbol{\omega}\times\boldsymbol{\upsilon} \quad \text{(show!)}$$

where $\upsilon = \|\boldsymbol{\upsilon}\|$ is the **velocity magnitude** and $\boldsymbol{\omega} = \nabla\times\boldsymbol{\upsilon}$ is the rotation of the velocity field called **vorticity**. The proof of the identity $(\boldsymbol{\upsilon}\cdot\nabla)\boldsymbol{\upsilon} = \nabla\left(\tfrac{1}{2}\upsilon^2\right) + \boldsymbol{\omega}\times\boldsymbol{\upsilon}$ is recommended as the exercise for the Reader.