PROGRAM ROZWOJOWY
POLITECHNIKI WARSZAWSKIEJ

LECTURE 3

GENERAL FRAMEWORK FOR CONSERVATION LAWS IN FLUID MECHANICS. THE PRINCIPLE OF MASS CONSERVATION AND RELATED EQUATIONS.

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Fundamental Principles of Mechanics tell us what happens with:

- **mass**
- **linear momentum**
- **angular momentum**
- **energy**

during a motion of a fluid medium.

Basic equations of the Fluid Mechanics are derived from these principles.

Additionally, the reference to the **2nd Principle of Thermodynamics** may be necessary in order to recognized physically feasible solutions.

CONSERVATION LAWS – GENERAL FRAMEWORK

Consider an extensive physical quantity *H*. The spatial distribution of this quantity can be characterized by means of its mass-specific density *h*. AT this point we do not precise if the field *h* is scalar, vector or tensor.

Consider the finite control (not fluid!) volume Ω embedded in the fluid. The total amount of the quantity characterized by the density field h is expressed by the volume integral

$$H(t) = \int_{\Omega} \rho h \, dV$$

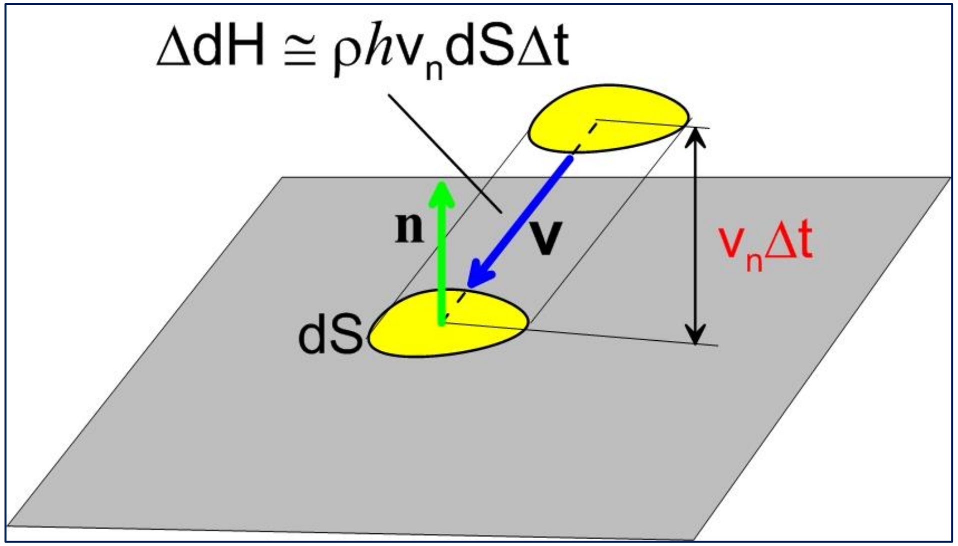
where ρ denotes the mass density of the fluid. We ask the fundamental question: what is the rate of the temporal change of *H*? The general answer is

$$\frac{dH}{dt} \equiv \frac{d}{dt} \int_{\Omega} \rho h \, dV = \left. \frac{dH}{dt} \right|_{production} + \left. \frac{dH}{dt} \right|_{flow\ through\ \partial\Omega}$$

i.e., The total rate is the sum of two contributions:

- change rate due to the production/destruction of the quantity *H*,
- change rate due to the transport of *H* by the fluid entering/leaving Ω through the boundary $\partial\Omega$.

Note that the second contribution can be expressed by the following surface integral (see figure)



$$\left.\frac{dH}{dt}\right|_{flow\ through\ \partial\Omega} = -\int_{\partial\Omega} \rho h \upsilon_n \, dS$$

where $\left.\upsilon_n\right|_{\partial\Omega} = \boldsymbol{\upsilon}\cdot\boldsymbol{n}\big|_{\partial\Omega}$ denotes the normal component of the fluid velocity at the boundary. The negative sign in the formula appears due to the fact that the normal vector **n** point outwards, so the negative value of υ_n corresponds to the incoming flow (positive – for the outflow).

The general principle of conservation (or rather variation!) of the quantity *H* can be cast into the following form

$$\left.\frac{dH}{dt}\right|_{production} = \mathcal{E}_{sources}$$

where $\mathcal{E}_{sources}$ stands for the "source" terms which describe time-specific production or destruction of the quantity H in the volume Ω.

The particular character of the source terms depends on the quantity H:

1. Mass of fluid

Then $h \equiv 1$ and

$$H \equiv M(t) = \int_{\Omega} \rho \, dV$$

In this case $\mathcal{E}_{sources} \equiv 0$ since mass cannot be produced or created!

2. Linear momentum

Then $h \equiv \boldsymbol{v}$ and

$$H \equiv \boldsymbol{P}(t) = \int_{\Omega} \rho \boldsymbol{v} \, dV$$

In this case the source term is the sum of all external forces acting on the fluid contained in Ω

$$\mathcal{E}_{sources} \equiv \underbrace{\boldsymbol{F}_S}_{\text{surface forces on } \partial\Omega} + \underbrace{\boldsymbol{F}_V}_{\text{volumetric forces in } \Omega} = \int_{\partial\Omega} \boldsymbol{\sigma} \, dS + \int_{\Omega} \rho \boldsymbol{f} \, dV$$

where $\boldsymbol{\sigma}$ denotes the stress vector at the boundary $\partial\Omega$.

3. Angular momentum

Then $h \equiv \boldsymbol{x} \times \boldsymbol{\upsilon}$ and

$$\boldsymbol{K}(t) = \int_{\Omega} \boldsymbol{x} \times \rho \boldsymbol{\upsilon} \, dV$$

In this case, the source term is the sum of all external moments of forces acting on the fluid contained in Ω

$$\mathcal{E}_{sources} \equiv \underbrace{\boldsymbol{M}_S}_{\substack{\text{surface} \\ \text{moment on } \partial\Omega}} + \underbrace{\boldsymbol{M}_V}_{\substack{\text{volumetric} \\ \text{moment in } \Omega}} = \int_{\partial\Omega} \boldsymbol{x} \times \boldsymbol{\sigma} \, dS + \int_{\Omega} \boldsymbol{x} \times \rho \boldsymbol{f} \, dV$$

4. Energy

Here we mean total energy which is the sum of internal and kinetic energy of the fluid.

Then $h \equiv e = u + \tfrac{1}{2} \boldsymbol{\upsilon} \cdot \boldsymbol{\upsilon} = u + \tfrac{1}{2} \upsilon^2$ and

$$H \equiv E(t) = \int_{\Omega} \rho (u + \tfrac{1}{2} \upsilon^2) \, dV$$

where u denotes the mass-specific internal energy of the fluid and υ is the magnitude of the fluid velocity.

The source terms include:

- work performed per one time unit (power) by surface and volumetric forces
- conductive heat transfer through the boundary $\partial\Omega$
- heat production by internal processes and /or by absorbed radiation.

We can write

$$\mathcal{E}_{sources}(t) = \underbrace{P_S + P_V}_{\text{power of external forces}} + \underbrace{Q_{\partial\Omega}}_{\text{conduction of heat through } \partial\Omega} + \underbrace{Q_{\Omega}}_{\text{internal heat sources}}$$

where the mechanical power terms are

$$P_S = \int_{\partial\Omega} \boldsymbol{\sigma} \cdot \boldsymbol{v} \, dS \quad , \quad P_V = \int_{\Omega} \rho \boldsymbol{f} \cdot \boldsymbol{v} \, dV$$

and the heat terms are

$$Q_{\partial\Omega} = -\int_{\partial\Omega} \boldsymbol{q}_h \cdot \boldsymbol{n} \, dS \quad , \quad Q_{\Omega} = \int_{\Omega} \rho \gamma_h \, dV$$

In the above, the symbol $\boldsymbol{q}_h$ denotes the vector of conductive heat flux through the boundary $\partial\Omega$ (we will see later that it can be expressed by the temperature gradient) and the symbol γ_h stands for the mass-specific density of internal heat sources in the fluid.

EQUATION OF MASS CONSERVATION

We have already mentioned that for the mass the source terms are absent. Thus, we have

$$\left.\frac{dM}{dt}\right|_{production} = \frac{dM}{dt} - \left.\frac{dM}{dt}\right|_{flow\ through\ \partial\Omega} = 0$$

or, equivalently

$$\frac{d}{dt}\int_{\Omega} \rho\, dV - \left(-\int_{\partial\Omega} \rho \boldsymbol{v}\cdot\boldsymbol{n}\, dS\right) = 0$$

Since the volume Ω is fixed we can change order of the volume integration and time differentiation. We can also apply the GGO Theorem to the surface integral to transform it to the volume one. This is what we get

$$\int_{\Omega} \left[\frac{\partial}{\partial t}\rho + \nabla\cdot(\rho\boldsymbol{v})\right] dV = 0$$

Finally, since the volume Ω can be chosen as arbitrary part of the whole flow domain then – assuming sufficient regularity of the integrated expression – we conclude that

$$\boxed{\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho \boldsymbol{v}) = 0}$$

at each point of the fluid domain. We have derived the **differential equation of mass conservation**!

The obtained form of this equations is called **conservative** (sic!). However, other equivalent forms can be obtained by using standard manipulations with differential operators

$$0 = \frac{\partial}{\partial t}\rho + \nabla \cdot (\rho \boldsymbol{v}) = \underbrace{\frac{\partial}{\partial t}\rho + \boldsymbol{v} \cdot \nabla \rho}_{\frac{D}{Dt}\rho} + \rho \nabla \cdot \boldsymbol{v} = \frac{D}{Dt}\rho + \rho \nabla \cdot \boldsymbol{v}$$

In the index notation

$$0 = \frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x_j}(\rho \upsilon_j) = \underbrace{\frac{\partial}{\partial t}\rho + \upsilon_j \frac{\partial}{\partial x_j}\rho}_{\frac{D}{Dt}\rho} + \rho \frac{\partial}{\partial x_j}\upsilon_j = \frac{D}{Dt}\rho + \rho \frac{\partial}{\partial x_j}\upsilon_j$$

Note that:

1. If the flow is stationary, i.e. none of the parameters is explicitly time-dependent, then the equation of mass conservation reduces to the form

$$\nabla \cdot (\rho \boldsymbol{v}) = \boldsymbol{v} \cdot \nabla \rho + \rho \nabla \cdot \boldsymbol{v} = 0$$

2. If $\rho \equiv const$ then the mass conservation equation reduces to the particularly simple form (the continuity equation)

$$\nabla \cdot \boldsymbol{v} = 0$$

In words: **the divergence of the velocity field of the constant-density fluid (liquid) vanishes identically in the whole flow domain.** Note that this condition is the geometric constrain imposed on the class of admissible vector fields rather than evolutionary equation.

TWO-DIMENSIONAL INCOMPRESSIBLE FLOW. STREAMFUNCTION.

The streamfunction is a very convenient concept in the theory of **2D** incompressible flow. The idea is to introduce the **scalar field** $\psi = \psi(t, x_1, x_2)$ such that

$$\upsilon_1 = \partial_{x_2}\psi \qquad , \qquad \upsilon_2 = -\partial_{x_1}\psi$$

Note that the continuity equation (**see Lecture 3**)

$$\partial_{x_1}\upsilon_1 + \partial_{x_2}\upsilon_2 = 0$$

is satisfied automatically. Indeed, we have

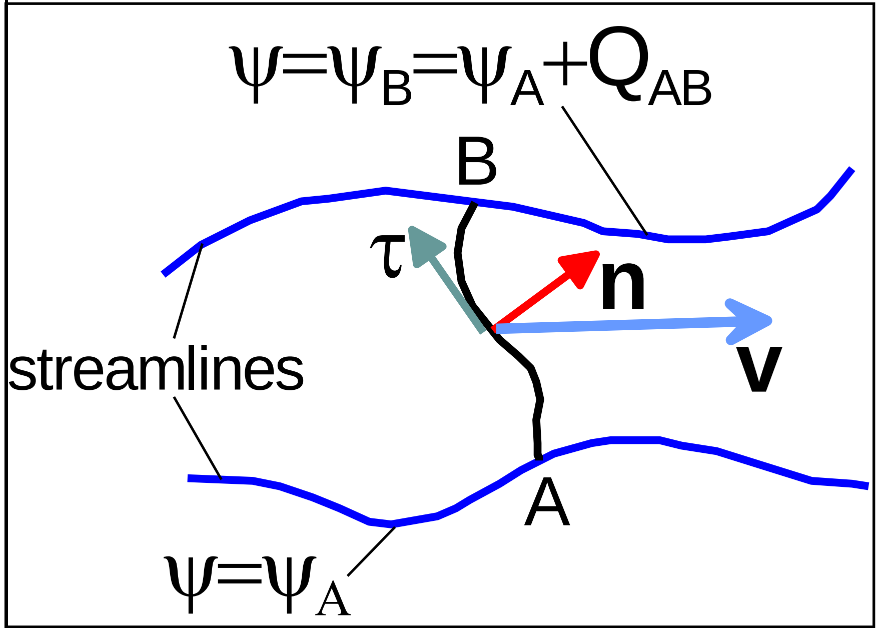
$$\partial_{x_1}\upsilon_1 + \partial_{x_2}\upsilon_2 = \partial_{x_1,x_2}\psi - \partial_{x_2,x_1}\psi = 0$$

The streamfunction has a remarkable property: it is constant along streamlines.

To see this, it is sufficient to show that the gradient of the streamfunction is always perpendicular to the velocity vector (why?). It is indeed the case:

$$\nabla\psi \cdot \boldsymbol{\upsilon} = \upsilon_1 \partial_{x_1}\psi + \upsilon_2 \partial_{x_2}\psi = -\upsilon_1\upsilon_2 + \upsilon_2\upsilon_1 = 0$$

Consider a line joining two points in the (plane) flow domain. We will calculate the volumetric flow rate (the volume flux) through this line.



We have

$$Q_{AB} = \int_A^B \boldsymbol{v}\cdot\boldsymbol{n}\,ds = \int_A^B (\upsilon_1 n_1 + \upsilon_2 n_2)\,ds =$$

$$= \int_A^B (\upsilon_1 \tau_2 - \upsilon_2 \tau_1)\,ds = \int_A^B (\tau_1 \partial_{x_1}\psi + \tau_2 \partial_{x_2}\psi)\,ds =$$

$$= \int_A^B \nabla\psi\cdot d\boldsymbol{s} = \psi_B - \psi_A$$

The volumetric flux through the line segment is equal to the difference of the streamfunction between the endpoints of this segment.

Note: the **scalar stream function** can also be defined for axisymmetric flows. In general 3D flows, the **vector stream function** $\boldsymbol{\Psi}$ can be introduced, such that $\boldsymbol{v} = \nabla\times\boldsymbol{\Psi}$. Note that this relation implies automatically that $\nabla\cdot\boldsymbol{v} = 0$, i.e. continuity equation is satisfied.