

LECTURE 12

ENTROPY AND THERMODYNAMIC INEQUALITY



ENTROPY OF A SMOOTH FLOW OF IDEAL FLUID

We will show that **if the flow is smooth (i.e., all kinematic and thermodynamic fields are sufficiently regular) then the entropy of the fluid is conserved along trajectories of fluid elements.**

To this end, let us consider the equation of internal energy derived in the Lecture 11. For the ideal fluid (no viscosity and no heat conduction) this equation reduces to

$$\rho \frac{D}{Dt} u = -p \nabla \cdot \mathbf{v}$$

We have already seen the relation

$$\nabla \cdot \mathbf{v} = -\frac{1}{\rho} \frac{D}{Dt} \rho$$

which is implied by the mass conservation equation.

Thus, the equation for the internal energy u can be written as follows

$$\frac{D}{Dt} u = -\frac{p}{\rho^2} \frac{D}{Dt} \rho = -p \frac{D}{Dt} (1/\rho) = -p \frac{D}{Dt} \mathcal{G}$$

Let us remind that the 1st Principle of Thermodynamics can be expressed in terms of complete differentials of three parameters of thermodynamic state: entropy s , internal energy u and specific volume $\mathcal{V} = 1/\rho$. The corresponding form of this principle reads

$$Tds = du + pd\mathcal{V}$$

For the thermodynamic process inside individual fluid element one can write

$$T \frac{D}{Dt}s = \frac{D}{Dt}u + p \frac{D}{Dt}\mathcal{V} = -p \frac{D}{Dt}\mathcal{V} + p \frac{D}{Dt}\mathcal{V} = 0$$

In the above, the equation for the internal energy has been used. We see that **entropy of the fluid is fixed along trajectories**, as stated. We will show later that this statement is no longer valid if strong discontinuities (called **shock waves**) appear.

We have already introduced the concept of **homoenergetic** flows. In such flows we have

$$i + \frac{1}{2}v^2 - \Phi = C_e^{global}, \quad \text{or equivalently} \quad \nabla(i + \frac{1}{2}v^2 - \Phi) = 0.$$

Similarly, we call the flow **homoentropic** if $\nabla s \equiv 0$. Thus, when the flow is homoentropic then the entropy is uniformly distributed in the flow domain.

Since the 1st Principle of Thermodynamics can be written in the following form

$$T ds = di - (1/\rho) dp$$

then for any stationary flow one has

$$T \nabla s = \nabla i - (1/\rho) \nabla p$$

In the case of a homoentropic flow we get

$$\nabla i = (1/\rho) \nabla p = \nabla P.$$

Thus, **if the flow is homoenergetic and homoentropic, it is automatically barotropic and the Bernoulli constant C_B is global.** Note that in the case of 2D flows, it implies that the velocity field is potential (explain why!).

Yet another interesting result can be derived from the Euler equation written in the Lamb-Gromeko form for the stationary flow

$$\nabla\left(\frac{1}{2}v^2\right) + \mathbf{v} \times \boldsymbol{\omega} = -\frac{1}{\rho} \nabla p + \nabla \Phi$$

Using the entropy/enthalpy form of the 1st thermodynamic principle, we can re-write the above equation in the following form called the **Crocco Equation**

$$T \nabla s = \nabla \left(\frac{1}{2} v^2 + i - \Phi \right) + v \times \omega$$

According to the **Crocco Equation**, any inhomogeneity in the spatial distribution of entropy in the homoenergetic flow immediately leads to vorticity generation.

THERMODYNAMIC INEQUALITY

Consider the control volume Ω . The change in time of the total entropy of the fluid which occupies instantaneously this volume can be expressed as the sum of three contributions:

- the change caused by the transport of entropy in or out of Ω through its boundary $\partial\Omega$,
- the change caused by the heat conduction (due to inhomogeneous temperature distribution),
- the change caused by thermodynamically irreversible processes of any sort inside Ω .

The **Second Principle of Thermodynamics** says that any thermodynamically irreversible process makes the entropy increasing.

In other words: **the total change rate of the entropy contained in the volume Ω must not be smaller than the change rate due to the entropy transport and heat conduction.**

As before, the total amount of entropy contained in the volume Ω at a given time instant can be expressed by the following volumetric integral

$$S_{\Omega} = \int_{\Omega} \rho s dV$$

Since Ω is fixed in time, the total rate of change of S_Ω is equal

$$\left. \frac{dS_\Omega}{dt} \right|_{total} = \int_\Omega \frac{\partial}{\partial t} (\rho s) dV$$

Using the formula derived in the **Lecture 3**, we can express the rate of change due to the convective transport of entropy through the boundary by the following integral

$$\left. \frac{dS_\Omega}{dt} \right|_{\substack{\text{transport} \\ \text{through } \partial\Omega}} = - \int_{\partial\Omega} \rho s \mathbf{v} \cdot \mathbf{n} dA$$

Thus, the net rate change of entropy in Ω is equal (see Lecture 3 for mathematical details of integration; note that the mass conservation equation is used to simplify the form of the final integral)

$$\begin{aligned} \left. \frac{dS_\Omega}{dt} \right|_{\text{production}} &= \left. \frac{dS_\Omega}{dt} \right|_{total} - \left. \frac{dS_\Omega}{dt} \right|_{\substack{\text{transport} \\ \text{through } \partial\Omega}} = \\ &= \int_\Omega \frac{\partial}{\partial t} (\rho s) dV + \int_{\partial\Omega} \rho s \mathbf{v} \cdot \mathbf{n} dA = \dots = \int_\Omega \rho \frac{D}{Dt} s dV \end{aligned}$$

Accordingly to the 2nd Principle of Thermodynamics, the entropy “production” rate is at least equal to the increase of entropy caused by heat conduction.

In order to evaluate the rate of entropy change due to heat conduction, consider the small portion $\Delta\Omega$ of the volume Ω . The boundary of this portion will be denoted by the symbol ΔA . The amount of heat exchanged by this volume during a small time interval Δt can be expressed as

$$\Delta Q = -\Delta t \int_{\Delta A} \mathbf{q} \cdot \mathbf{n} dA = -\Delta t \int_{\Delta\Omega} \nabla \cdot \mathbf{q} dV \approx -\Delta t \nabla \cdot \mathbf{q}(\mathbf{x}) \Delta V$$

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Theor.*

In the above, \mathbf{q} is the local heat flux vector and ΔV is the volume of $\Delta\Omega$. The corresponding entropy change in $\Delta\Omega$ in the time interval Δt is equal

$$\Delta S_{\Omega}|_{heat} = \frac{\Delta Q}{T} \approx -\Delta t \frac{\nabla \cdot \mathbf{q}(\mathbf{x})}{T} \Delta V$$

Integration over the whole volume Ω , division by Δt and taking the limit $\Delta t \rightarrow 0$ yields the formula

$$\left. \frac{dS_\Omega}{dt} \right|_{\text{heat}} = - \int_\Omega \frac{\nabla \cdot \mathbf{q}(\mathbf{x})}{T} dV$$

Then, the 2nd Principle of Thermodynamics implies that

$$\left. \frac{dS_\Omega}{dt} \right|_{\text{production}} \geq \left. \frac{dS_\Omega}{dt} \right|_{\text{heat}}$$

or

$$\int_\Omega \rho \frac{Ds}{Dt} dV \geq - \int_\Omega \frac{\nabla \cdot \mathbf{q}(\mathbf{x})}{T} dV$$

The heat flux integral term can be transformed as follows (explain!)

$$\int_\Omega \frac{\nabla \cdot \mathbf{q}}{T} dV = \int_\Omega \nabla \cdot \left(\frac{\mathbf{q}}{T} \right) dV + \int_\Omega \frac{\mathbf{q} \cdot \nabla T}{T^2} dV = \int_{\partial\Omega} \frac{\mathbf{q} \cdot \mathbf{n}}{T} dV + \int_\Omega \frac{\mathbf{q} \cdot \nabla T}{T^2} dV$$

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Thus the above **thermodynamic inequality** can be written as

$$\int_{\Omega} \rho \frac{Ds}{Dt} dV \geq - \int_{\partial\Omega} \frac{\mathbf{q} \cdot \mathbf{n}}{T} dA - \int_{\Omega} \frac{\mathbf{q} \cdot \nabla T}{T^2} dV$$

For a **thermally isotropic fluid**, the heat flux is parallel (and oppositely oriented) to the local temperature gradient (the **Fourier Law**)

$$\mathbf{q} = -\lambda \nabla T$$

where $\lambda > 0$ is the coefficient of heat transfer.

Inserting the Fourier form of the heat flux into the thermodynamic inequality we obtain the final form (**the Gibbs-Duhem inequality**)

$$\int_{\Omega} \rho \frac{Ds}{Dt} dV > \int_{\partial\Omega} \lambda \frac{\nabla T \cdot \mathbf{n}}{T} dA + \int_{\Omega} \lambda \left| \frac{\nabla T}{T} \right|^2 d\Omega$$

The interpretation can be formulated as follows: if the heat transfer is present in the flow then there exists a minimal admissible rate of entropy production.