BEAMS
RITZ-RAYLAYGH METHOD and FINITE ELEMENT METHOD

Principle of minimum potential energy.

The **potential energy** of an elastic body is defined as

Total potential energy (V)= Strain energy (U) – potential energy of loading (Wz)

In theory of elasticity the potential energy is the sum of the elastic energy and the work potential:

$$V = U - W_z = \frac{1}{2}\int_{\Omega} \sigma_{ij}\varepsilon_{ij} d\Omega - \int_{\Omega} X_i u_i d\Omega - \int_{\Gamma} p_i u_i d\Gamma$$

Ω – domain of the elastic body, Γ – boundary, σ_{ij} – stress state tensor, ε_{ij} – strain state tensor ,

u_i – displacement vector, p_i – boundary load (traction), X_i– body loads

The potential energy is a functional of the displacement field. The body force is prescribed over the volume of the body, and the traction is prescribed on the surface Γ. The first two integral extends over the volume of the body. The third integral extends over the boundary.

The principle of minimum potential energy states that,

the displacement field that represents the solution of the problem fullfills the displacement boundary conditions and inimizes the total potential energy.

$$V = U - W_z = \min!,$$

Total potential energy of the beam loaded by the distributed load $p\left[\frac{\mathrm{N}}{\mathrm{m}}\right]$:

$$V=\frac{1}{2}\int_0^l EI(w'')^2\,dx-\int_0^l pw\,dx\,,$$

where the function w(x) describes deflection of the beam

Ritz method

1.The problem must be stated in a variational form, as a minimization problem, that is:
find w(x) minimizing the functional V(w)

2. The solution is approximated by a finite linear combination of the form:

$$\tilde{w}(x)=\sum_{i=1}^{n} a_i\varphi_i(x)$$

where a_i denote the undetermined parameters termed **Ritz coefficients**,
and φi are the assumed **approximation functions** (i=1,2...n).
The approximate functions φi must be linearly independent and

3. Finally functional V is approximated by the function of n variables a_i

$V=V(a_1, a_2, a_3, ...a_n)$

3. The parameters a_i are determined by requirement that the functional is minimized with respect to a_i

$$\frac{\partial V}{\partial a_i}=0\,,\qquad i=1,\dots,n\,.$$

$$[A]\{a\}=\{b\}$$

4. The solution provides a_i, and the approximate solution

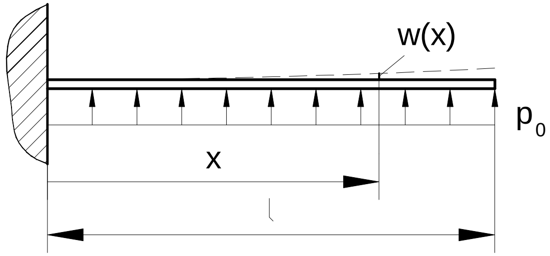
$$\tilde{w}(x)=\sum_{i=1}^{n} a_i\varphi_i(x)\,.$$

Hence the approximate internal forces in the beam

$$\tilde{M}_q(x)=EI\tilde{w}''(x),$$
$$\tilde{T}(x)=-EI\tilde{w}'''(x).$$

EXAMPLE

Find the deflection of the cantilever beam under the load $p_0\left[\frac{N}{m}\right]$ using the analytical solution of the differential equation and compare it to the approximate solution using Ritz method with the function $\tilde{w}(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3$.



Exact analytical solution

$$w''(x) = \frac{M_g(x)}{EI} \qquad M_q(x) = \frac{p_0}{2}(l-x)^2,$$

$$w(x) = 0, \quad d\frac{w(x)}{dx} = 0$$

Solution

$$w(x) = \frac{p_0}{24EI}(6l^2 - 4lx + x^2)x^2,$$

Max. deflection w(l)= p_0 l^2 /8 EI

The approximate solution $\tilde{w}(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3$ has to satisfy the displacement boundary conditions

$$\tilde{w}(x=0) = 0, \qquad \tilde{w}'(x=0) = 0\,.$$

Thus

$$\tilde{w}(x) = a_3 x^2 + a_4 x^3.$$

$$V = \frac{EI}{2}(4a_3^2 l + 12 a_3 a_4 l^2 + 12 a_4^2 l^3) - p(a_3\frac{l^3}{3} + a_4\frac{l^4}{4})\,.$$

$$\frac{\partial V}{\partial a_3} = \frac{EI}{2}\left(8la_3 + 12l^2 a_4\right) - \frac{p_0 l^3}{3} = 0,$$

$$\frac{\partial V}{\partial a_4} = \frac{EI}{2}\left(12l^2 a_3 + 24l^3 a_4\right) - \frac{p_0 l^4}{4} = 0.$$

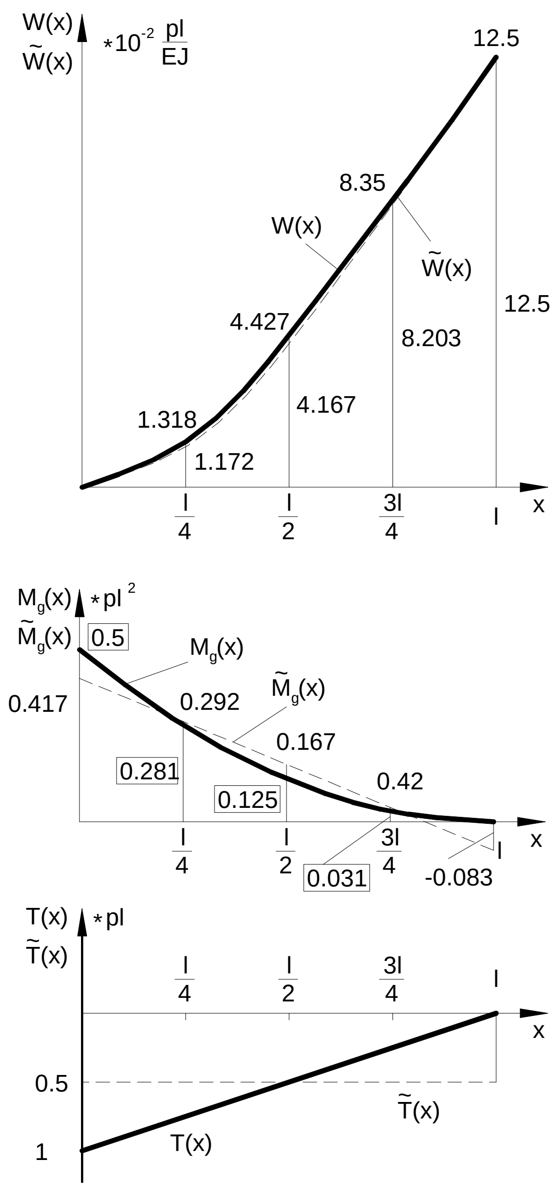
$$a_3 = \frac{5}{24}\frac{p_0 l^2}{EI}, \quad a_4 = -\frac{p_0 l}{12EI}$$

Finally the approximate solution is

$$\tilde{w}(x) = \frac{5}{24}\frac{p_0 l^2}{EI}x^2 - \frac{p_0}{12EI}x^3,$$

$$\tilde{M}_q(x) = \frac{5}{12}p_0 l^2 - \frac{p_0 l}{2}x,$$

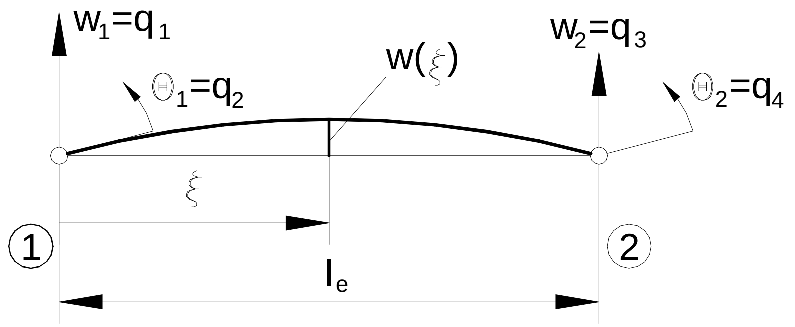
$$\tilde{T}(x) = \frac{-p_0 l}{2}.$$



Graphs presenting exact (bold line) and approximate (dashed line) solutions of the cantilever beam:
displacement, bending moment, shear force

Finite Element Method approach

Approximation : local, with nodal displacements w_1 , w_2 , θ_1 *and* θ_2 as unknown parameters



Positive directions:
upward for translation
counter clockwise for rotation

Simple beam finite element

Lets assume first the polynomial approximation within the element

$$w(\xi) = \alpha_1 + \alpha_2\xi + \alpha_3\xi^2 + \alpha_4\xi^3$$

with four unknown parameters α_i .

The required new parameters : nodal displacements w_1 , w_2 , θ_1 *and* θ_2 (degrees of freedom – DOF of the element)

Nodal displacement vector

$$\{q\}_e = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_e = \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} .$$

$$w(\xi) = \sum_{i=1}^{4} N_i(\xi) q_i \, .$$

$$w(\xi) = \lfloor N(\xi) \rfloor \{q\}_e \, ,$$

Relation between $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and q_1, q_2, q_3, q_4

$$q_1 = w(0) = \alpha_1,$$ displacement and node 1

$$q_2 = \frac{dw}{d\xi}(0) = \alpha_2,$$ slope at node 1

$$q_3 = w(l) = \alpha_1 + \alpha_2 l_e + \alpha_3 l_e^2 + \alpha_4 l_e^3,$$ displacement at node 2

$$q_4 = \frac{dw}{d\xi}(l) = \alpha_2 + 2\alpha_3 l_e + 3\alpha_4 l_e^2.$$ slope at node 2

In the matrix form

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l_e & l_e^2 & l_e^3 \\ 0 & 1 & 2l_e & 3l_e^2 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix}.$$

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-3}{l_e^2} & \frac{-2}{l} & \frac{3}{l_e^2} & \frac{-1}{l_e} \\ \frac{2}{l_e^3} & \frac{1}{l_e} & \frac{-2}{l_e^3} & \frac{1}{l_e^2} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_e$$

The approximate deflection may be presented in the form

$$w(\xi) = \lfloor 1, \xi, \xi^2, \xi^3 \rfloor \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = \lfloor N_1(\xi), N_2(\xi), N_3(\xi), N_4(\xi) \rfloor \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix},$$

$$N_1(\xi) = 1 - 3\frac{\xi^2}{l_e^2} + 2\frac{\xi^3}{l_e^3},$$

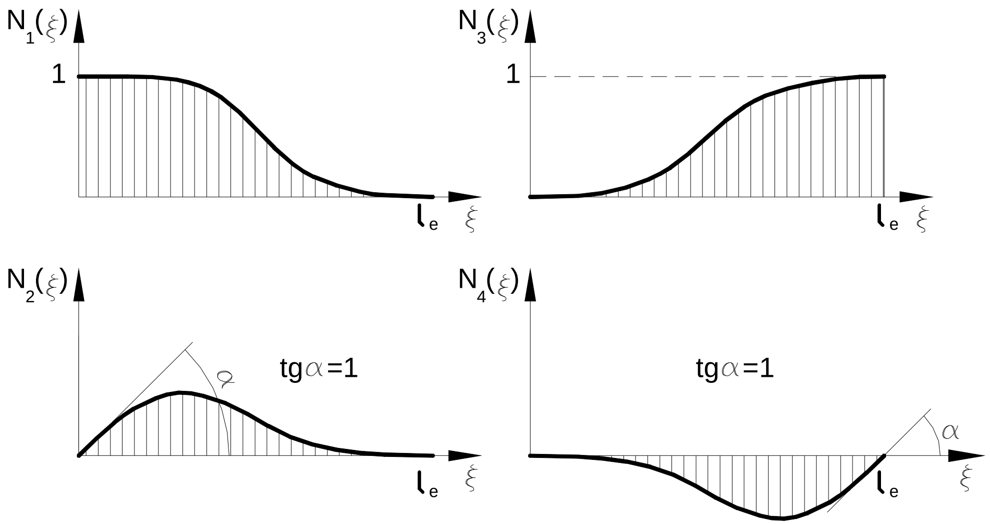
$$N_2(\xi) = \xi - 2\frac{\xi^2}{l_e} + \frac{\xi^3}{l_e^2},$$

$$N_3(\xi) = 3\frac{\xi^2}{l_e^2} - 2\frac{\xi^3}{l_e^3},$$

$$N_4(\xi) = \frac{-\xi^2}{l_e} + \frac{\xi^3}{l_e^2}.$$

The functions $N_i(\xi)$ are called **shape functions** of the beam element.

$N_i(\xi)$ describes deflection of the beam element , where $q_i = 1$, and for $j \neq i \quad q_j = 0$ (see graphs).



Shape functions of a beam element

$$w(\xi) = \lfloor N(\xi) \rfloor \{q\}_e ,$$
$$w'(\xi) = \lfloor N'(\xi) \rfloor \{q\}_e ,$$
$$w''(\xi) = \lfloor N''(\xi) \rfloor \{q\}_e .$$

Total potential energy of the beam element of the length l_e

$$V_e = U_e - W_{ze} = \frac{EI}{2} \int_0^{l_e} (w''(\xi))^2 d\xi - \int_0^{l_e} p(\xi) w(\xi) d\xi .$$

$$U_e = \frac{EI}{2}\int_0^{l_e} w''(\xi)w''(\xi)d\xi = \frac{EI}{2}\int_0^{l_e} \lfloor q \rfloor_e \{N''\}\lfloor N'' \rfloor\{q\}_e \, d\xi =$$

$$= \frac{EI}{2}\lfloor q \rfloor_e \int_0^{l_e} \begin{bmatrix} N_1''N_1'' & N_1''N_2'' & N_1''N_3'' & N_1''N_4'' \\ N_2''N_1'' & N_2''N_2'' & N_2''N_3'' & N_2''N_4'' \\ N_3''N_1'' & N_3''N_2'' & N_3''N_3'' & N_3''N_4'' \\ N_4''N_1'' & N_4''N_2'' & N_4''N_3'' & N_4''N_4'' \end{bmatrix} d\xi \{q\}_e .$$

$$U_e = \frac{1}{2}\lfloor q \rfloor_e [k]_e \{q\}_e ,$$

$$[k]_e = EI \begin{bmatrix} \int_0^{l_e} N_1''N_1''d\xi & \int_0^{l_e} N_1''N_2''d\xi & \int_0^{l_e} N_1''N_3''d\xi & \int_0^{l_e} N_1''N_4''d\xi \\ \int_0^{l_e} N_2''N_1''d\xi & \int_0^{l_e} N_2''N_2''d\xi & \int_0^{l_e} N_2''N_3''d\xi & \int_0^{l_e} N_2''N_4''d\xi \\ \int_0^{l_e} N_3''N_1''d\xi & \int_0^{l_e} N_3''N_2''d\xi & \int_0^{l_e} N_3''N_3''d\xi & \int_0^{l_e} N_3''N_4''d\xi \\ \int_0^{l_e} N_4''N_1''d\xi & \int_0^{l_e} N_4''N_2''d\xi & \int_0^{l_e} N_4''N_3''d\xi & \int_0^{l_e} N_4''N_4''d\xi \end{bmatrix}$$

Matrix $[k]_e$ is named stiffness matrix of beam element. After integration

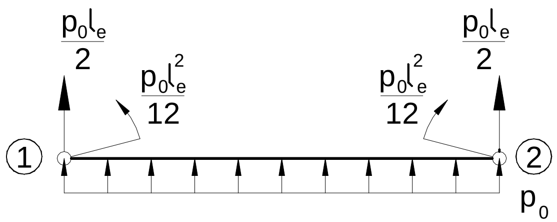
$$[k]_e = \frac{2EI}{l_e^3}\begin{bmatrix} 6 & 3l_e & -6 & 3l_e \\ 3l_e & 2l_e^2 & -3l_e & l_e^2 \\ -6 & -3l_e & 6 & -3l_e \\ 3l_e & l_e^2 & -3l_e & 2l_e^2 \end{bmatrix}.$$

The external work done by the traction p:

$$W_{ze}^{p} = \int_{0}^{l_e} p(\xi) w(\xi) d\xi = \int_{0}^{l_e} p(\xi) \lfloor N(\xi) \rfloor \{q\}_e \, d\xi = \int_{0}^{l_e} \lfloor N_1(\xi) p(\xi) d\xi, N_2(\xi) p(\xi) d\xi, N_3(\xi) p(\xi) d\xi, N_4(\xi) p(\xi) d\xi \rfloor \{q\}_e \, d\xi,$$

$$W_{ze}^{p} = \lfloor F_1^e, F_2^e, F_3^e, F_4^e \rfloor_e \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \lfloor F \rfloor_e \{q\}_e, \qquad F_i^e = \int_{0}^{l_e} N_i(\xi) p(\xi) d\xi$$

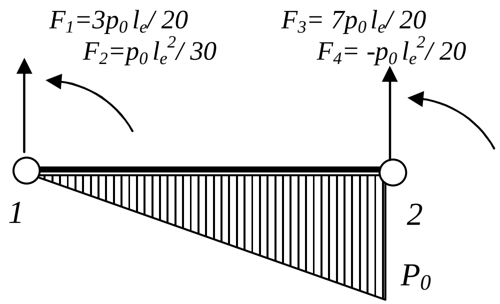
F_i^e - equivalent nodal forces



$$F_1^e = F_3^e = \frac{p_0 l_e}{2}$$

$$F_2^e = \frac{p_0 l_e^2}{12}$$

$$F_4^e = \frac{-p_0 l_e^2}{12}$$



Equivalent nodal forces corresponding to the constant and linear distribution of p_0 load
(kinematically equivalent or work-equivalent !)

Total potential energy of the beam element

$$V_e = U_e - W_{ze} = \frac{1}{2} \underset{1\times4}{\lfloor q \rfloor_e} \underset{4\times4}{[k]_e} \underset{4\times1}{\{q\}_e} - \underset{1\times4}{\lfloor q \rfloor_e} \underset{4\times1}{\{F\}_e} .$$

The conditions for finding the minimum of V_e:

$$\frac{\partial V_e}{\partial q_i} = 0 , \qquad i = 1, 2, 3, \ldots, n$$

$$[k]_e \{q\}_e = \{F\}_e . \qquad \frac{2EI}{l_e^3} \begin{bmatrix} 6 & 3l_e & -6 & 3l_e \\ 3l_e & 2l_e^2 & -3l_e & l_e^2 \\ -6 & -3l_e & 6 & -3l_e \\ 3l_e & l_e^2 & -3l_e & 2l_e^2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_e = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}_e$$

Set of linear equations for one element model of the considered cantilever beam:

$$\frac{2EI}{l^3}\begin{bmatrix} 6 & 3l & -6 & 3l \\ 3l & 2l^2 & -3l & l^2 \\ -6 & -3l & 6 & -3l \\ 3l & l^2 & -3l & 2l^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ \frac{p_0 l}{2} \\ \frac{-p_0 l^2}{12} \end{Bmatrix}$$

Constraints $q_1=0$ and $q_2=0$ may be taken into account by

the transformation of the set of equation to the form $[A]\begin{Bmatrix} F_1 \\ F_2 \\ q_3 \\ q_4 \end{Bmatrix} = \{b\}$ or by reductionof the problem to

$$\frac{2EI}{l^3}(6q_3 - 3lq_4) = \frac{p_0 l}{2},$$

$$\frac{2EI}{l^3}(-3lq_3 + 2l^2 q_4) = \frac{-p_0 l^2}{12},$$

Solution is:

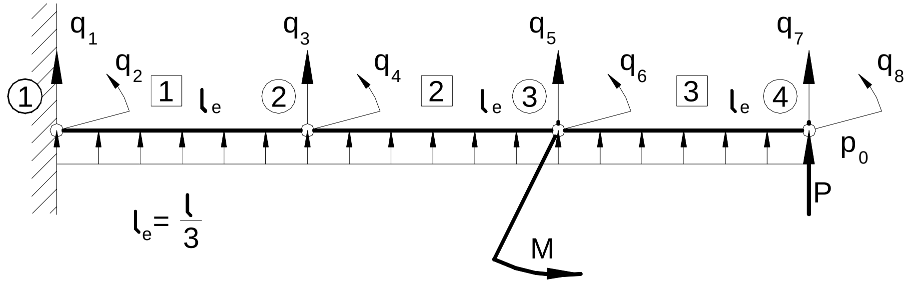
$$q_3 = \frac{1}{8}\frac{p_0 l^4}{EI}$$

$$q_4 = \frac{1}{6}\frac{p_0 l^3}{EI}$$

Finally the deflection function from the one element model is $w(\xi) = \left(\frac{3}{8} - \frac{1}{6}\right)\frac{p_0 l^2}{EI}\xi^2 + \left(\frac{-2}{8} + \frac{1}{6}\right)\frac{p_0 l}{EI}\xi^3 = \frac{5}{24}\frac{p_0 l^2}{EI}\xi^2 - \frac{p_0 l}{12EI}\xi^3$

The same result as obtained in the case of Ritz method – why?

Dividing the beam into LE elements

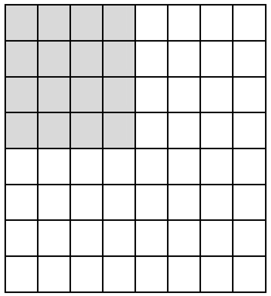


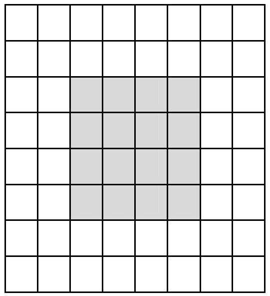
global nodal displacements vector $\{q\} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{Bmatrix} = \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \\ w_4 \\ \theta_4 \end{Bmatrix}.$

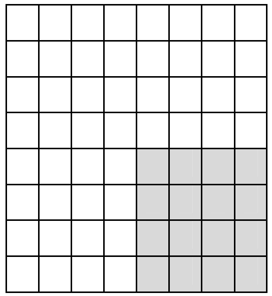
N=8 nodal diplacements (degrees of freedom of the FE model)

Strain energy U_e of each of the elements

$$U_e = \frac{1}{2} \underset{1x4}{\lfloor q \rfloor_e} \underset{4x4}{[k]_e} \underset{4x1}{\{q\}_e} \qquad = \frac{1}{2} \underset{1\times N}{\lfloor q \rfloor} \underset{N\times N}{[k^*]_e} \underset{N\times 1}{\{q\}},$$

$[k^*]_1 =$ 

$[k^*]_2 =$ 

$[k^*]_3 =$ 

element 1 with the global DOF : q_1, q_2, q_3, q_4

element 2 with the global DOF : q_3, q_4, q_5, q_6

element 3 with the global DOF : q_5, q_6, q_7, q_8

$$U = \sum_{e=1}^{LE} U_e = \frac{1}{2}\lfloor q \rfloor \left(\sum_{i=1}^{LE} \left[k^* \right]_e \right) \{q\} = \frac{1}{2} \lfloor q \rfloor [K] \{q\} .$$

$$V = U - W_z = \frac{1}{2}\lfloor q \rfloor [K]\{q\} - \lfloor q \rfloor \{F\} ,$$

$$\frac{\partial V}{\partial q_i} = 0 , \qquad i = 1, 2, 3, \ldots, n$$

$$[K]\{q\} = \{F\}$$. + displacement boundary conditions (constraints)

For each element the internal forces M,T are calculated separately:

$$M_q(\xi) = EIw''(\xi) = EI \left\lfloor N_1'', N_2'', N_3'', N_4'' \right\rfloor \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_e ,$$

$$M_q(\xi) = \left[\frac{12}{l_e^3}(\xi - \frac{l_e}{2}) q_1 + \frac{6}{l_e^2}(\xi - \frac{2}{3} l_e) q_2 - \frac{12}{l_e^3}(\xi - \frac{l_e}{2}) q_3 + \frac{6}{l_e^2}(\xi - \frac{l_e}{3}) q_4 \right] EI ,$$

$$T(\xi) = -EIw'''(\xi) = EI \left\lfloor N_1''', N_2''', N_3''', N_4''' \right\rfloor \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_e .$$

$$T(\xi) = -\left[\frac{12}{l_e^3}(q_1 - q_3) + \frac{6}{l_e^2}(q_2 + q_4) \right] EI .$$

For the case of 3-element model shown in the figure the final set of linear equations is

$$
\begin{bmatrix}
k_{11}^1 & k_{12}^1 & k_{13}^1 & k_{14}^1 & 0 & 0 & 0 & 0 \\
k_{21}^1 & k_{22}^1 & k_{23}^1 & k_{24}^1 & 0 & 0 & 0 & 0 \\
k_{31}^1 & k_{32}^1 & k_{33}^1 + k_{11}^2 & k_{34}^1 + k_{12}^2 & k_{13}^2 & k_{14}^2 & 0 & 0 \\
k_{41}^1 & k_{42}^1 & k_{43}^1 + k_{21}^2 & k_{44}^1 + k_{22}^2 & k_{23}^2 & k_{24}^2 & 0 & 0 \\
0 & 0 & k_{31}^2 & k_{32}^2 & k_{33}^2 + k_{11}^3 & k_{34}^2 + k_{12}^3 & k_{13}^3 & k_{14}^3 \\
0 & 0 & k_{41}^2 & k_{42}^2 & k_{43}^2 + k_{21}^3 & k_{44}^2 + k_{22}^3 & k_{23}^3 & k_{24}^3 \\
0 & 0 & 0 & 0 & k_{31}^3 & k_{32}^3 & k_{33}^3 & k_{34}^3 \\
0 & 0 & 0 & 0 & k_{41}^3 & k_{42}^3 & k_{43}^3 & k_{44}^3
\end{bmatrix}
\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{Bmatrix}
=
\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{Bmatrix}
$$

$$
\frac{2EI}{l_e^3}
\begin{bmatrix}
6 & 3l_e & -6 & 3l_e & 0 & 0 & 0 & 0 \\
3l_e & 2l_e^2 & -3l_e & l_e^2 & 0 & 0 & 0 & 0 \\
-6 & -3l_e & 12 & 0 & -6 & 3l_e & 0 & 0 \\
3l_e & l_e^2 & 0 & 4l_e^2 & -3l_e & l_e^2 & 0 & 0 \\
0 & 0 & -6 & -3l_e & 12 & 0 & -6 & 3l_e \\
0 & 0 & 3l_e & l_e^2 & 0 & 4l_e^2 & -3l_e & l_e^2 \\
0 & 0 & 0 & 0 & -6 & -3l_e & 6 & -3l_e \\
0 & 0 & 0 & 0 & 3l_e & l_e^2 & -3l_e & 2l_e^2
\end{bmatrix}
\begin{Bmatrix} 0 \\ 0 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{Bmatrix}
=
\begin{Bmatrix} F_1 \\ F_2 \\ p_0 l_e \\ 0 \\ p_0 l_e \\ M \\ P + \dfrac{p_0 l_e}{2} \\ \dfrac{-p_0 l_e^2}{12} \end{Bmatrix}
$$

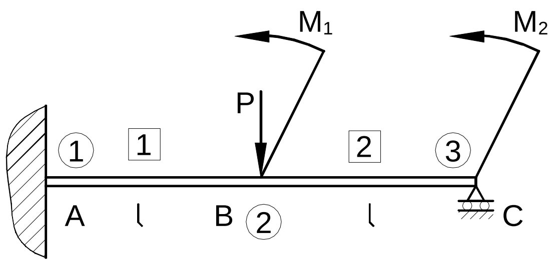
FEM calculations:

1. Generation of stiffness matrices $\underset{4x4}{[k]_e}$ for all elements
2. Assembling the element matrices to obtain the global stiffness matrix $\underset{NxN}{[K]}$
3. Finding the equivalent nodal force vector $\underset{Nx1}{\{F\}}$
4. Imposing the boundary conditions and the solution of the final set of linear equations – finding all nodal displacements $\underset{Nx1}{\{q\}}$
5. Calculation of the internal forces (bending moment, shear force) and the stresses within the elements

The example

Final set of equations (3 active DOF)

$$\frac{2EI}{l^3}\begin{bmatrix} 12 & 0 & 3l \\ 0 & 4l^2 & l^2 \\ 3l & l^2 & 2l^2 \end{bmatrix}\begin{Bmatrix} q_3 \\ q_4 \\ q_6 \end{Bmatrix} = \begin{Bmatrix} -P \\ M_1 \\ M_2 \end{Bmatrix}.$$



$$\begin{Bmatrix} q_3 \\ q_4 \\ q_6 \end{Bmatrix} = \begin{Bmatrix} w_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \frac{l}{96EI}\begin{bmatrix} 7l^2 & 3l & -12l \\ 3l & 15 & -12 \\ -12l & -12 & 48 \end{bmatrix}\begin{Bmatrix} -P \\ M_1 \\ M_2 \end{Bmatrix}.$$

(exact solution – why?)