BARS AND SPRINGS

Finite element of a bar under axial loads:

Assuming nodal displacements u_1 i u_2 we have $u(\xi)$ as the linear function: $u(\xi) = u_1 + \frac{u_2 - u_1}{l_e}\xi$.

After some operations $u(\xi)$ may be presented in the standard form as dependent on the nodal displacements and shape functions:

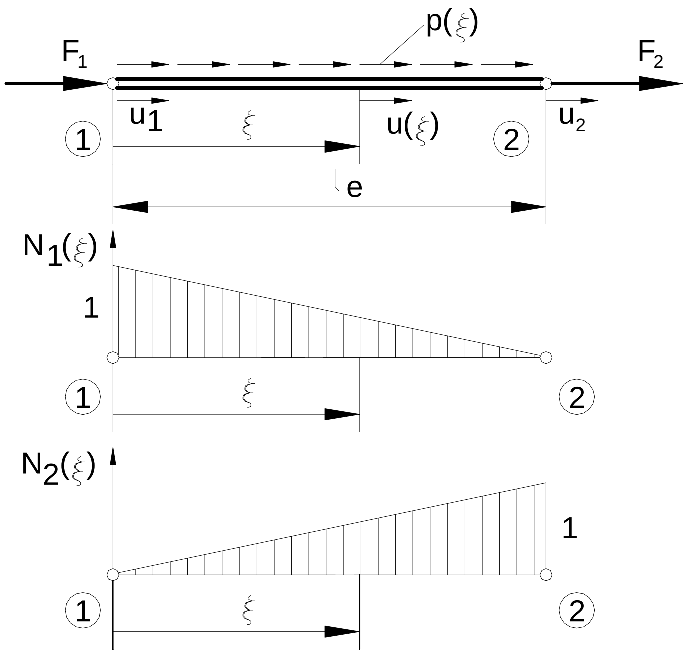
$$u(\xi) = \left(1 - \frac{\xi}{l}\right)u_1 + \frac{\xi}{l}u_2 = \lfloor N_1(\xi), N_2(\xi) \rfloor \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e = \lfloor N \rfloor \{q\}_e,$$

where

$$\{q\}_e = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_e$$ is the vector of nodal displacements

$$\lfloor N \rfloor = \lfloor N_1(\xi), N_2(\xi) \rfloor$$ is the vector of shape functions

$$N_1(\xi) = 1 - \frac{\xi}{l_e}, \qquad N_2(\xi) = \frac{\xi}{l_e},$$



Tension bar element with 2 nodes and 2 degrees of freedom and its shape functions

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Strain energy of the element:

$$U_e = \frac{1}{2} A \int_0^{l_e} \sigma(\xi)\varepsilon(\xi) d\xi = \frac{EA}{2} \int_0^{l_e} (\varepsilon(\xi))^2 d\xi .$$

Taking into account that

$$\varepsilon(\xi) = \frac{du}{d\xi} = \lfloor N_1', N_2' \rfloor \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e .$$

we have

$$U_e = \frac{EA}{2} \int_0^{l_e} \lfloor q_1, q_2 \rfloor_e \begin{Bmatrix} N_1' \\ N_2' \end{Bmatrix} \lfloor N_1', N_2' \rfloor \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e d\xi =$$

$$= \frac{EA}{2} \lfloor q_1, q_2 \rfloor_e \int_0^{l_e} \begin{bmatrix} N_1' N_1' & N_1' N_2' \\ N_2' N_1' & N_2' N_2' \end{bmatrix} d\xi \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e = \frac{1}{2} \lfloor q \rfloor_e [k]_e \{q\}_e ,$$

where

$$[k]_e = \frac{EA}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

is the stiffness matrix of the rod element (symmetric, singular, positive semidefinite)

Equivalent nodal forces

The forces equivalent to the distributed load $p(\xi)\left[\frac{\mathrm{N}}{\mathrm{m}}\right]$.

$$W_{ze}^{p}=\int_0^{l_e} p(\xi)u(\xi)d\xi=\int_0^{l_e}\lfloor N_1(\xi)\,p(\xi),N_2(\xi)\,p(\xi)\rfloor\begin{Bmatrix}q_1\\q_2\end{Bmatrix}_e d\xi=$$

$$=\left\lfloor \int_0^{l_e} N_1(\xi)\,p(\xi),\int_0^{l_e} N_2(\xi)\,p(\xi)d\xi \right\rfloor\begin{Bmatrix}q_1\\q_2\end{Bmatrix}_e .$$

In result:

$$W_{ze}^{p}=\lfloor F_1^e,F_2^e\rfloor_e\begin{Bmatrix}q_1\\q_2\end{Bmatrix}_e, \quad \text{where} \qquad F_i^e=\int_0^{l_e} N_i(\xi)p(\xi)d\xi ,$$

F_i^e - the nodal forces equivalent to the distributed load p (‘work-equivalent’ or ‘kinematically’ equivalent)

Next steps of FE modelling are similar as in the case of the beam element. Finally we get the system of linear quations :

$$[K]\{q\}=\{F\}.$$

The right side vector $\{F\}$ contains the external forces acting on nodes of the model (active nodes and reactions).

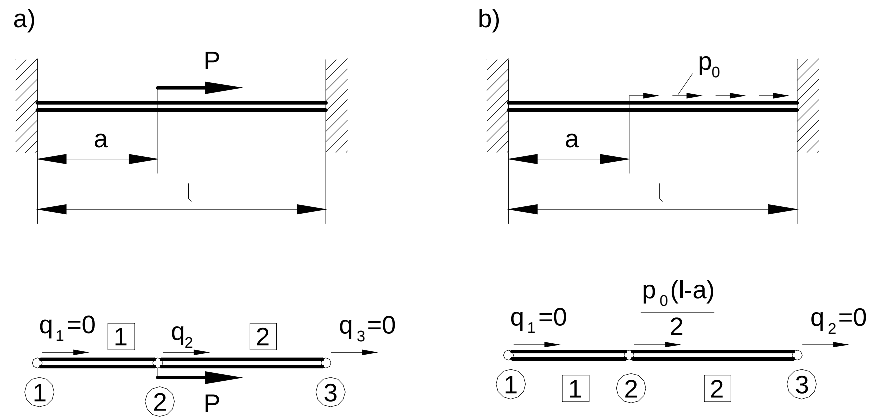
The system is solved after taking into account all boundary conditions;

When the vector of nodal displacements is determined the stresses within each of elements are computed:

$$\sigma=E\varepsilon=E\left\lfloor N_1'(\xi),N_2'(\xi)\right\rfloor\begin{Bmatrix}q_1\\q_2\end{Bmatrix}_e=\frac{E(q_2-q_1)}{l_e}.$$

Example.

Solve the presented below rods using FE models consisted of 2 elements



Stiffness matrices of the two finite elements

$$[k]_e^1 = \frac{EA}{a}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad [k]_e^2 = \frac{EA}{l-a}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

System of simultaneous linear equations

$$EA\begin{bmatrix} \frac{1}{a} & -\frac{1}{a} & 0 \\ -\frac{1}{a} & \frac{1}{a}+\frac{1}{l-a} & -\frac{1}{l-a} \\ & -\frac{1}{l-a} & \frac{1}{l-a} \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

.

After including the boundary conditions $q_1 = q_3 = 0$ and $F_2 = P$ (<u>case a)</u>) we have

$$q_2 = \frac{P(l-a)a}{EAl},$$

$$F_1 = \frac{-P(l-a)}{l},$$

$$F_3 = \frac{-Pa}{l}.$$

where F_1 and F_3 are the nodal forces (reactions).

<u>In the case b</u> the nodal force in the second node is:

$$F_2 = \frac{p_0(l-a)}{2},$$

Thus $q_2 = \frac{p_0(l-a)^2 a}{2lEA}$, $F_1 = \frac{-p_0(l-a)^2}{2l}$, $F_3 = \frac{-p_0 a(l-a)}{2l}$.

The reaction in the first node $R_1 = F_1$

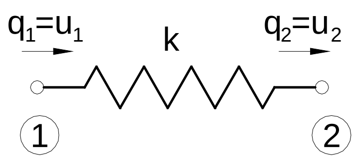
And the reaction in the third node

$$R_3 = F_3 - \frac{p_0(l-a)l}{2l} = \frac{-p_0 a(l-a)}{2l} - \frac{p_0(l-a)l}{2l} = \frac{-p_0(l-a)(l+a)}{2l}.$$

$$R_1 + R_3 = -p_0(l-a).$$

FE solution in the case a is the exact one but in the case b the approximate (why?)

Spring element



$$F=k*\Delta u=k(u_2-u_1)$$

$$[k]_e=k\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

Finite element of a spring

Strain energy

$$U_e=\frac{1}{2}F\Delta u=\frac{1}{2}k(\Delta u)^2=\frac{1}{2}k(u_2-u_1)(u_2-u_1)\,.$$

$$U_e=\frac{1}{2}\lfloor u_1,u_2\rfloor\begin{bmatrix}k & -k\\ -k & k\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix},$$

$$U_e=\frac{1}{2}\lfloor q\rfloor_e[k]_e\{q\}_e\,,$$

$$[k]_e=\begin{bmatrix}k & -k\\ -k & k\end{bmatrix},$$ (stiffness matrix of a spring)

In the same way may be derived the stiffness matrix for the twisted shaft:

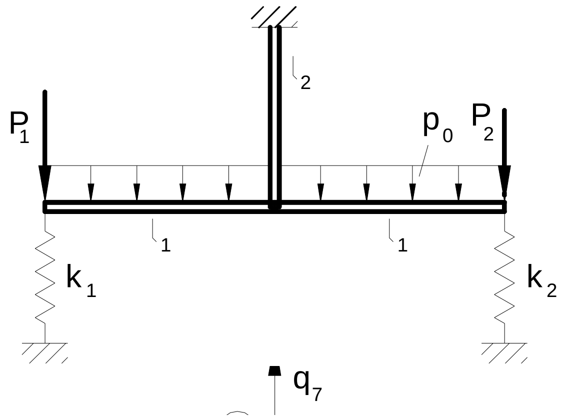
$$[k]_e=\frac{GI_s}{l_e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix},$$

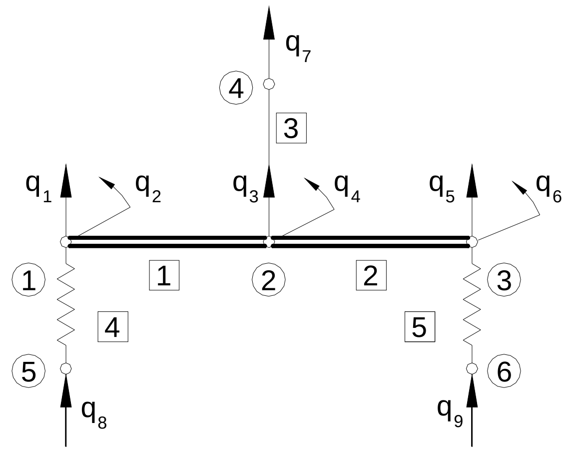
where GI_s is a torsional stiffness and the nodal displacements correspond to the rotation of the end cross-sections.

The FE models of the elastic structures can be built dividing the structure into finite elements of different types (beams, tension bars, springs etc.)

Example:

Find the finie element system of equations $[K]\{q\}=\{F\}$ for the structure presented below





Solution

FE model may be created using 2 beam elements , one rod element and 2 spring elements. The total number of degrees of freedom is 9

The stiffness matrices of the beam elements

$$[k]_e^1=[k]_e^2=\frac{2EI}{l_1^3}\begin{bmatrix} 6 & 3l_1 & -6 & 3l_1 \\ 3l_1 & 2l_1^2 & -3l_1 & l_1^2 \\ -6 & -3l_1 & 6 & -3l_1^2 \\ 3l_1 & l_1^2 & -3l_1 & 2l_1^2 \end{bmatrix}.$$

Degrees of freedom of the first element are q_1,q_2,q_3,q_4, and for the second q_3,q_4,q_5,q_6.

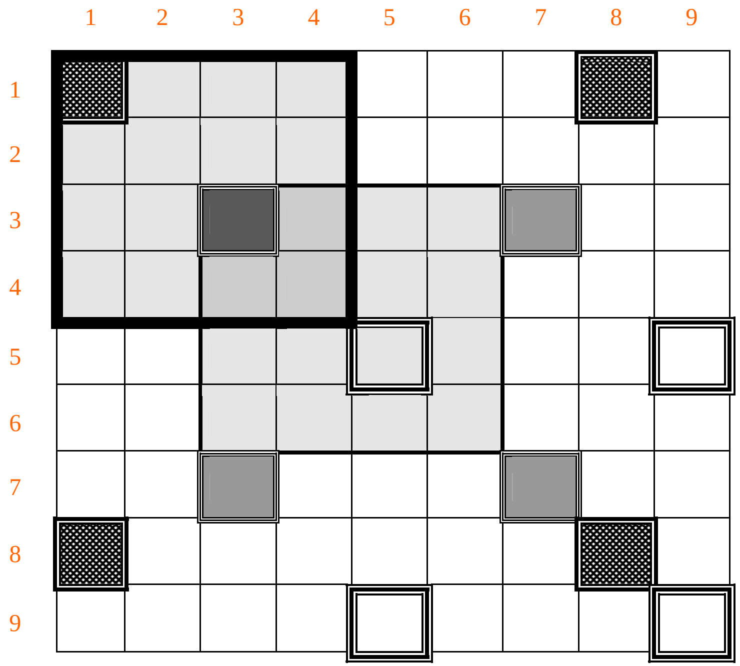
The stiffness matrix of the rod element is

$$[k]_e^3=\frac{EA}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Its degrees of fredom are q_3 and q_7.
The stiffness matrices of the springs:

$$[k]_e^4 = k_1\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad [k]_e^5 = k_2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

and corresponding degrees of freedom are q_8, q_1 and $q_9\ q_5$.
The FE system of equations $[K]\{q\}=\{F\}$ for the assuming numbering of the degrees of freedom:



$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \frac{-p_0 l_1}{2} - P_1 \\ \frac{-p_0 l_1^2}{12} \\ p_0 l_1 \\ 0 \\ \frac{-p_0 l_1}{2} - P_2 \\ \frac{p_0 l_1^2}{12} \\ F_7 \\ F_8 \\ F_9 \end{Bmatrix}$$

– Coefficients of the stiffness matrix of the element No 1 (beam)

– Coefficients of the stiffness matrix of the element No 2 (beam)

– Coefficients of the stiffness matrix of the element No 3 (rod)

– Coefficients of the stiffness matrix of the element No 4 (spring)

– Coefficients of the stiffness matrix of the element No 5 (spring)

$[K]$ may be written in the form

$$\underset{9\times 9}{[K]} = \begin{bmatrix}
k_{11}^{1}+k_{22}^{4} & k_{12}^{1} & k_{13}^{1} & k_{14}^{1} & 0 & 0 & 0 & k_{12}^{4} & 0 \\
k_{21}^{1} & k_{22}^{1} & k_{23}^{1} & k_{24}^{1} & 0 & 0 & 0 & 0 & 0 \\
k_{31}^{1} & k_{32}^{1} & k_{33}^{1}+k_{11}^{2}+k_{11}^{3} & k_{34}^{1}+k_{12}^{2} & k_{13}^{2} & k_{14}^{2} & k_{12}^{3} & 0 & 0 \\
k_{41}^{1} & k_{42}^{1} & k_{43}^{1}+k_{21}^{2} & k_{44}^{1}+k_{22}^{2} & k_{23}^{2} & k_{24}^{2} & 0 & 0 & 0 \\
0 & 0 & k_{31}^{2} & k_{32}^{2} & k_{33}^{2}+k_{22}^{5} & k_{34}^{2} & 0 & 0 & k_{12}^{5} \\
0 & 0 & k_{41}^{2} & k_{42}^{2} & k_{43}^{2} & k_{44}^{2} & 0 & 0 & 0 \\
0 & 0 & k_{21}^{3} & 0 & 0 & 0 & k_{11}^{3} & 0 & 0 \\
k_{21}^{4} & 0 & 0 & 0 & 0 & 0 & 0 & k_{11}^{4} & 0 \\
0 & 0 & 0 & 0 & k_{21}^{5} & 0 & 0 & 0 & k_{11}^{5}
\end{bmatrix}$$