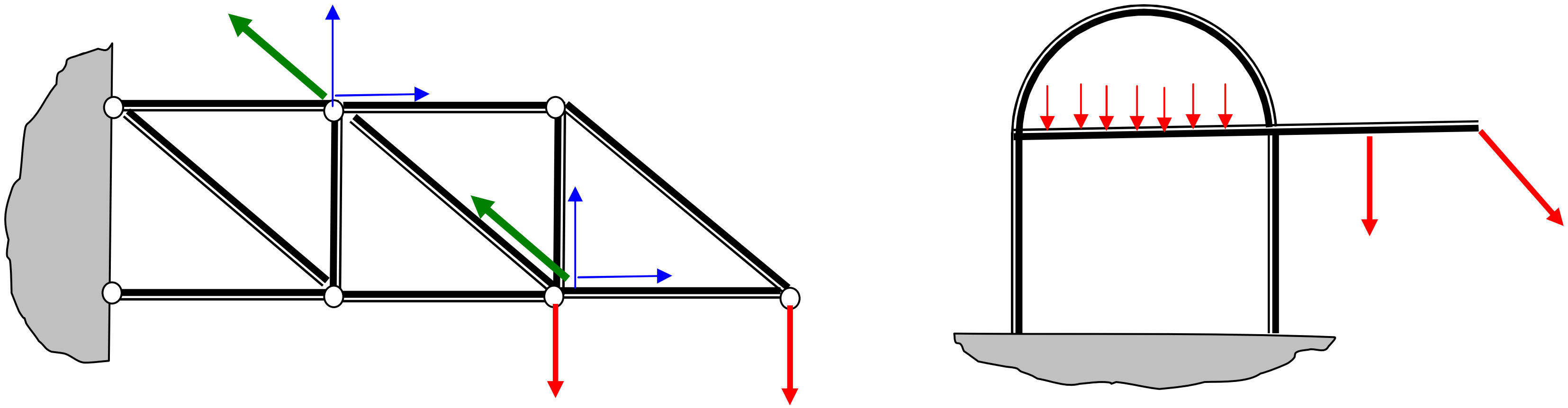
TRUSSES AND FRAMES

Trusses - structures made of simple straight bars (members), joined at their ends (nodes).

External forces and reactions to those forces are considered to act only at the nodes and result in forces in the members which are either tensile or compressive forces. Other internal forces are explicitly excluded because all the joints in a truss are treated as articulated joints.



The examples of 2D truss and 2D frame

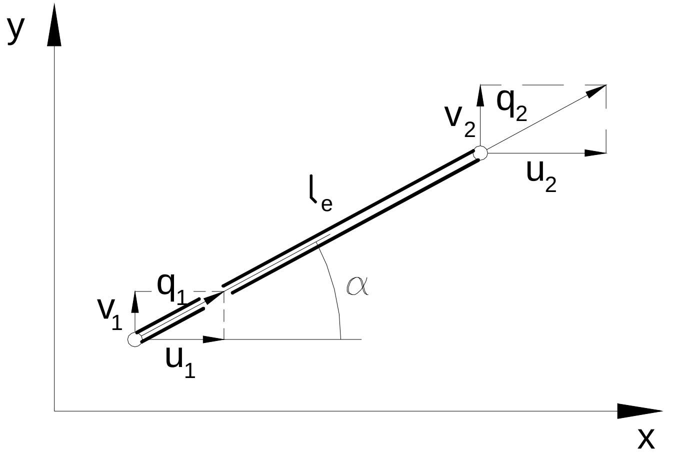
Frames are the structures with members that are rigidly connected - e.g. with welded joints. The members of frames can be loaded by concentrated and distributed forces. As a result they carry all possible internal forces (normal and shear forces, bending moments and torsinal moments).

TRUSSES

2D trusses

Relation between the nodal displacements in local (element) coordinate systems and in global coordinates

$\lfloor q \rfloor_e = \lfloor q_1, q_2 \rfloor_e$ along the axis of the rod $\lfloor q_g \rfloor_e = \lfloor u_1, \upsilon_1, u_2, \upsilon_2 \rfloor_e$ in x,y coordinate system



Finite element of a plane truss

$$q_i = u_i \cos\alpha + \upsilon_1 \sin\alpha \,. \quad (i = 1,2)$$

$$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \end{bmatrix} \begin{Bmatrix} u_1 \\ \upsilon_1 \\ u_2 \\ \upsilon_2 \end{Bmatrix}_e ,$$

$$\{q\}_e = [T_k]\{q_q\}_e$$

Strain energy of the element

$$U_e = \frac{1}{2} \underset{1\times2}{\lfloor q \rfloor} \underset{2\times2}{[k]_e} \underset{2\times1}{\{q\}_e} = \frac{1}{2} \underset{1\times4}{\lfloor q_q \rfloor_e^T} \underset{4\times2}{[T_k]^T} \underset{2\times2}{[k]_e} \underset{2\times4}{[T_k]} \underset{4\times1}{\{q_q\}_e}$$

$$U_e = \frac{1}{2} \lfloor q_q \rfloor_e [k_g]_e \{q_q\}_e ,$$

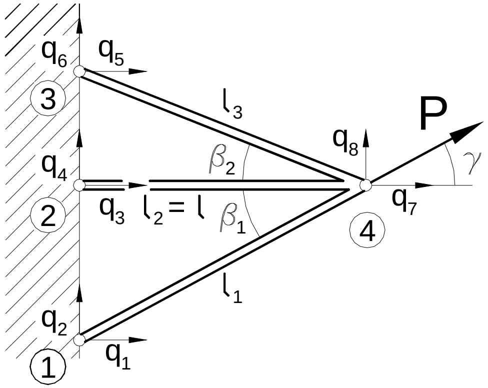
$$[k_g]_e = \frac{EA}{l_e} \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix} \qquad (*)$$

$$s = \sin\alpha, \; c = \cos\alpha$$

The stiffness matrix of the truss element in global coordinate system

Example.

Find the displacement vector of the node 4 of the simple 2D truss for the case $\beta_1 = \beta_2$ and the horizontal force P $(\gamma = 0)$.



Rozwiązanie

Element 1 nodes 1 and 4 slope angle $\alpha_1 = \beta_1$ length $l_1 = \dfrac{l}{\cos\alpha_1}$.

Element 2 nodes 2 and 4 slope angle $\alpha_2 = 0$ length $l_2 = \dfrac{l}{\cos\alpha_2}$.

Element 3 nodes 3 and 4 slope angle $\alpha_3 = -\beta_2$ length $l_3 = \dfrac{l}{\cos\alpha_3}$.

.The stiffness matrices of the three elements $\left[k_{ij}\right]_e^1, \left[k_{ij}\right]_e^2, \left[k_{ij}\right]_e^3$ are defined by (*).

The system of FE equations:

$$\begin{bmatrix}
k_{11}^1 & k_{12}^1 & 0 & 0 & 0 & 0 & k_{13}^1 & k_{14}^1 \\
k_{21}^1 & k_{22}^1 & 0 & 0 & 0 & 0 & k_{23}^1 & k_{24}^1 \\
0 & 0 & k_{11}^2 & k_{12}^2 & 0 & 0 & k_{13}^2 & k_{14}^2 \\
0 & 0 & k_{21}^2 & k_{22}^2 & 0 & 0 & k_{23}^2 & k_{24}^2 \\
0 & 0 & 0 & 0 & k_{11}^3 & k_{12}^3 & k_{13}^3 & k_{14}^3 \\
0 & 0 & 0 & 0 & k_{21}^3 & k_{22}^3 & k_{23}^3 & k_{24}^3 \\
k_{31}^1 & k_{32}^1 & k_{31}^2 & k_{32}^2 & k_{31}^3 & k_{32}^3 & k_{33}^1 + k_{33}^2 + k_{33}^3 & k_{34}^1 + k_{34}^2 + k_{34}^3 \\
k_{41}^1 & k_{42}^1 & k_{41}^2 & k_{42}^2 & k_{41}^3 & k_{42}^3 & k_{43}^1 + k_{43}^2 + k_{43}^3 & k_{44}^1 + k_{44}^2 + k_{44}^3
\end{bmatrix}
\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ q_7 \\ q_8 \end{Bmatrix}
=
\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ P\cos\gamma \\ P\sin\gamma \end{Bmatrix}.$$

Taking into account that $q_j = 0$ for $j = 1,6$ the set of quations may be reduced to

$$EA\left[\begin{array}{c|c} \sum_{i=1}^{3}\frac{c_i^2}{l_i} & \sum_{i=1}^{3}\frac{s_i c_i}{l_i} \\ \hline \sum_{i=1}^{3}\frac{s_i c_i}{l_i} & \sum_{i=1}^{3}\frac{s_i^2}{l_i} \end{array}\right]\begin{Bmatrix} q_7 \\ q_8 \end{Bmatrix} = \begin{Bmatrix} P\sin\gamma \\ P\cos\gamma \end{Bmatrix}.$$

Assuming $\beta_1 = \beta_2 = \beta \qquad \gamma = 0$

$$\frac{EA}{l}\left[\begin{array}{c|c} 1+2c^3 & 0 \\ \hline 0 & 2s^2c \end{array}\right]\begin{Bmatrix} q_7 \\ q_8 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix},$$

where $c = \cos\beta$, $s = \sin\beta$.

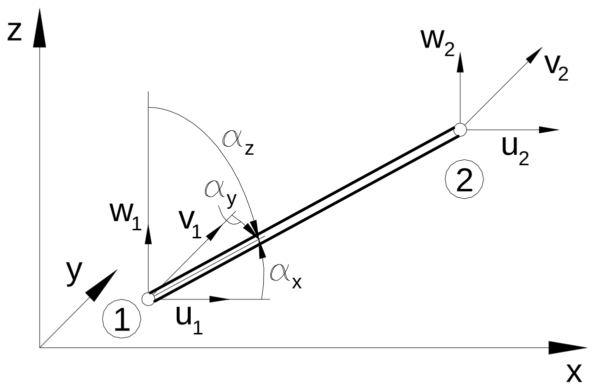
Then

$$q_7 = \frac{Pl}{EA(1+2c^3)},$$

$$q_8 = 0.$$

The normal forces in the elements are calculated from the nodal displacements in the local (element) coordinate systems

3D truss element in the coordinate system x,y,z



$$\{q\}_e = \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \end{Bmatrix}$$

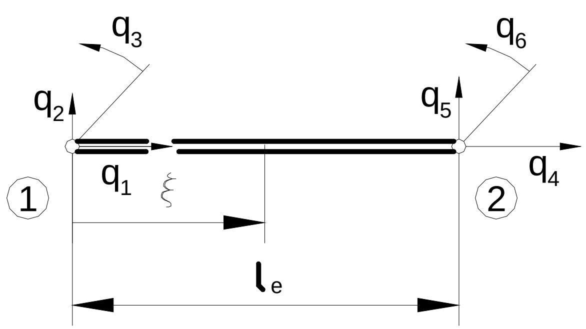
$$\left[k^g\right]_e = \frac{EA}{l_e} \begin{bmatrix} c_x^2 & c_x c_y & c_x c_z & -c_x^2 & -c_x c_y & -c_x c_z \\ c_x c_y & c_y^2 & c_y c_z & -c_x c_y & -c_y^2 & -c_y c_z \\ c_x c_z & c_y c_z & c_z^2 & -c_x c_z & -c_y c_z & -c_z^2 \\ -c_x^2 & -c_x c_y & -c_x c_z & c_x^2 & c_x c_y & c_x c_z \\ -c_x c_y & -c_y^2 & -c_y c_z & c_x c_y & c_y^2 & c_y c_z \\ -c_x c_z & -c_y c_z & -c_z^2 & c_x c_z & c_y c_z & c_z^2 \end{bmatrix}$$

where $c_x = \cos\alpha_x$, $c_y = \cos\alpha_y$, $c_z = \cos\alpha_z$.

FRAMES

2D frame element in the local coordinate system

The stiffness matrix of a frame element assembled from the stiffness matrices of the beam element with four degrees of freedom and the rod element with 2 degrees of freedom:



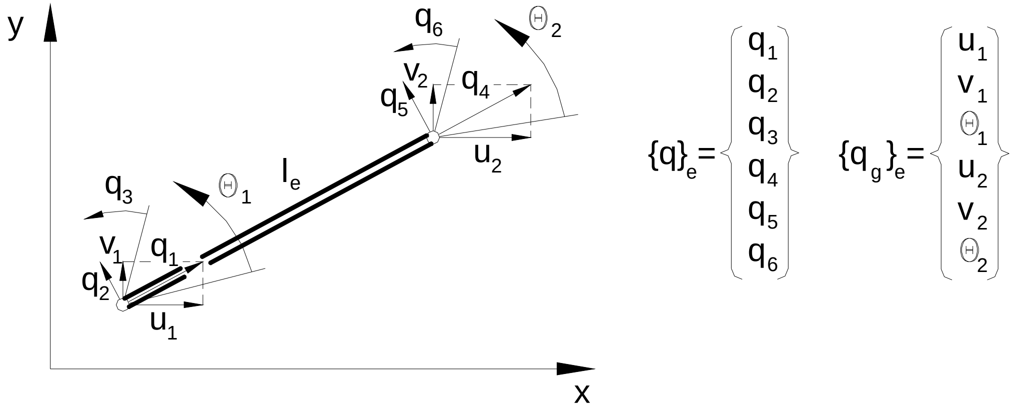
$$
\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix}
$$

$$
[k]_e = \begin{bmatrix}
\frac{EA}{l_e} & 0 & 0 & -\frac{EA}{l_e} & 0 & 0 \\
0 & \frac{12EI}{l_e^3} & \frac{6EI}{l_e^2} & 0 & \frac{-12EI}{l_e^3} & \frac{6EI}{l_e^2} \\
0 & \frac{6EI}{l_e^2} & \frac{4EI}{l_e} & 0 & \frac{-6EI}{l_e^2} & \frac{2EI}{l_e} \\
\frac{-EA}{l_e} & 0 & 0 & \frac{EA}{l_e} & 0 & 0 \\
0 & \frac{-12EI}{l_e^3} & \frac{-6EI}{l_e^2} & 0 & \frac{12EI}{l_e^3} & \frac{-6EI}{l_e^2} \\
0 & \frac{6EI}{l_e^2} & \frac{2EI}{l_e} & 0 & \frac{-6EI}{l_e^2} & \frac{4EI}{l_e}
\end{bmatrix}
$$

The stiffness matrix of a frame element in local coordinate system

The element with 6 DOF , the deformation defined by the functions $u(\xi)$ i $w(\xi)$ in the local c.s. It is called also 2D beam element.

2D frame element in the global coordinate system xy



The vectors of DOF of the frame element in the local c.s. $\{q\}_e$ *and in the global c.s.* $\{q_g\}_e$

The relation between the displacement of a node 1 in local (element) coordinate system and in global coordinate system

$$\begin{cases} q_1 = u_1 \cos\alpha + \upsilon_1 \sin\alpha, \\ q_2 = -u_1 \sin\alpha + \upsilon_1 \cos\alpha, \\ q_3 = \theta_1. \end{cases}$$

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}_e = [T_r] \begin{Bmatrix} u_1 \\ \upsilon_1 \\ \theta_1 \\ u_2 \\ \upsilon_2 \\ \theta_2 \end{Bmatrix} = [T_r] \cdot \{q_g\}_e ,$$

where the transformation matrix $\left[T_r\right]$ is

$$\left[T_r\right]=\begin{bmatrix} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Strain energy of the element

$$U_e=\frac{1}{2}\lfloor q \rfloor_e \left[k\right]_e \left\{q\right\}_e=\frac{1}{2}\lfloor q_g \rfloor_e \left[T_r\right]^T \left[k\right]_e \left[T_r\right]\left\{q_g\right\}_e,$$

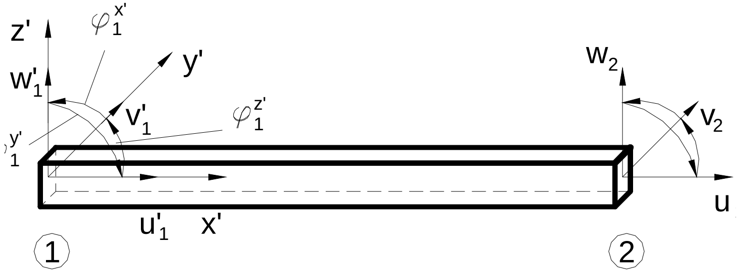
$$U_e=\frac{1}{2}\lfloor q_g \rfloor_e \left[k^g\right]_e \left\{q_g\right\}_e,$$

where

$$\left[k^g\right]_e=\left[T_r\right]^T \left[k\right]_e \left[T\right],$$

is the stiffness matrix of the 3D frame element in global c.s.

3D frames (beams)



$$\{q\}_e = \lfloor u'_1, v'_1, w'_1, \varphi_1^{x'}, \varphi_1^{y'}, \varphi_1^{z'}, u'_2, v'_2, w'_2, \varphi_2^{x'}, \varphi_2^{y'}, \varphi_2^{z'} \rfloor^T$$

The local (element) coordinate system is connected with the axis of the element.

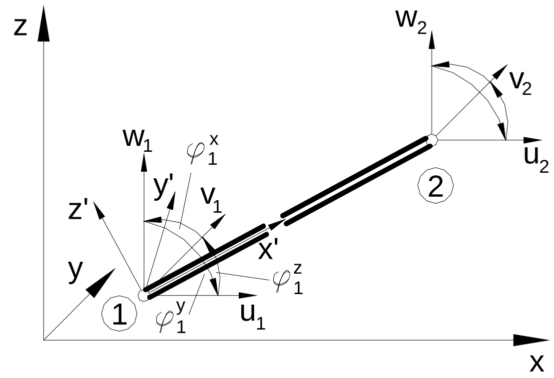
The element x'is oriented along the element.

The y' axis is automatically set parallel to the global xy plane

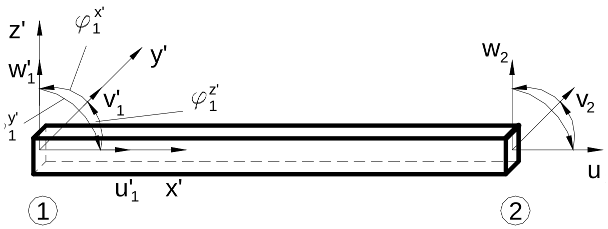
(If the element is perpendicular to the xy plane the x' is located in parralel to the global y axis)

The element input data include:

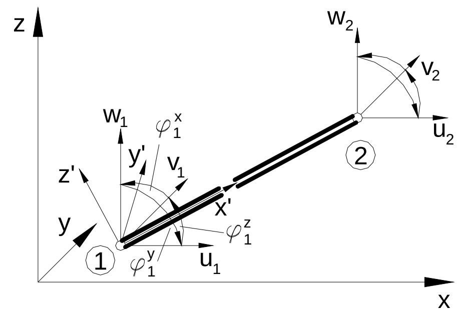
- the node locations
- the cross-sectional area
- 2 moments of inertia about the principal axes of the section
- the parameters defining shear stiffness in the principal directions and the torsional stiffness



$$\{q\}_e = \lfloor u_1, v_1, w_1, \varphi_1^{x}, \varphi_1^{y}, \varphi_1^{z}, u_2, v_2, w_2, \varphi_2^{x}, \varphi_2^{y}, \varphi_2^{z} \rfloor^T$$



$$\{q\}_e = \lfloor u'_1, v'_1, w'_1, \varphi_1^{x'}, \varphi_1^{y'}, \varphi_1^{z'}, u'_2, v'_2, w'_2, \varphi_2^{x'}, \varphi_2^{y'}, \varphi_2^{z'} \rfloor^T$$



$$\{q\}_e = \lfloor u_1, v_1, w_1, \varphi_1^{x}, \varphi_1^{y}, \varphi_1^{z}, u_2, v_2, w_2, \varphi_2^{x}, \varphi_2^{y}, \varphi_2^{z} \rfloor^T$$

$$[k]_e = \begin{bmatrix}
\frac{EA}{l_e} & & & & & & & & & & \text{macierz} & \text{symetryczna} \\
0 & \frac{12EI_{z'}}{l_e^3} & & & & & & & & & & \\
0 & 0 & \frac{12EI_{y'}}{l_e^3} & & & & & & & & & \\
0 & 0 & 0 & \frac{GI_s}{l_e} & & & & & & & & \\
0 & 0 & -\frac{6EI_{y'}}{l_e^2} & 0 & \frac{4EI_{y'}}{l_e} & & & & & & & \\
0 & \frac{6EI_{z'}}{l_e^2} & 0 & 0 & 0 & \frac{4EI_{z'}}{l_e} & & & & & & \\
-\frac{EA}{l_e} & 0 & 0 & 0 & 0 & 0 & \frac{EA}{l_e} & & & & & \\
0 & -\frac{12EI_{z'}}{l_e^3} & 0 & 0 & 0 & -\frac{6EI_{z'}}{l_e^2} & 0 & \frac{12EI_{z'}}{l_e^3} & & & & \\
0 & 0 & -\frac{12EI_{y'}}{l_e^3} & 0 & \frac{6EI_{z'}}{l_e^2} & 0 & 0 & 0 & \frac{12EI_{y'}}{l_e^3} & & & \\
0 & 0 & 0 & -\frac{GI_s}{l_e} & 0 & 0 & 0 & 0 & 0 & \frac{GI_s}{l_e} & & \\
0 & 0 & -\frac{6EI_{y'}}{l_e^2} & 0 & \frac{12EI_{z'}}{l} & 0 & 0 & 0 & \frac{6EI_{y'}}{l_e^2} & 0 & \frac{4EI_{y'}}{l_e} & \\
0 & \frac{6EI_{z'}}{l_e^2} & 0 & 0 & 0 & \frac{2EI_{z'}}{l_e^2} & 0 & -\frac{6EI_{z'}}{l_e^2} & 0 & 0 & 0 & \frac{4EI_{z'}}{l_e}
\end{bmatrix}$$

3D beam element and the corresponding stiffness matrix in the local (element) coordinate system $x'y'z'$