

# *Wytrzymałość konstrukcji 1*

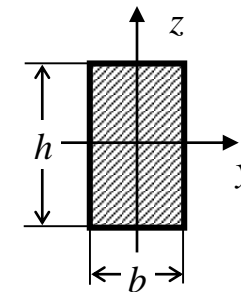
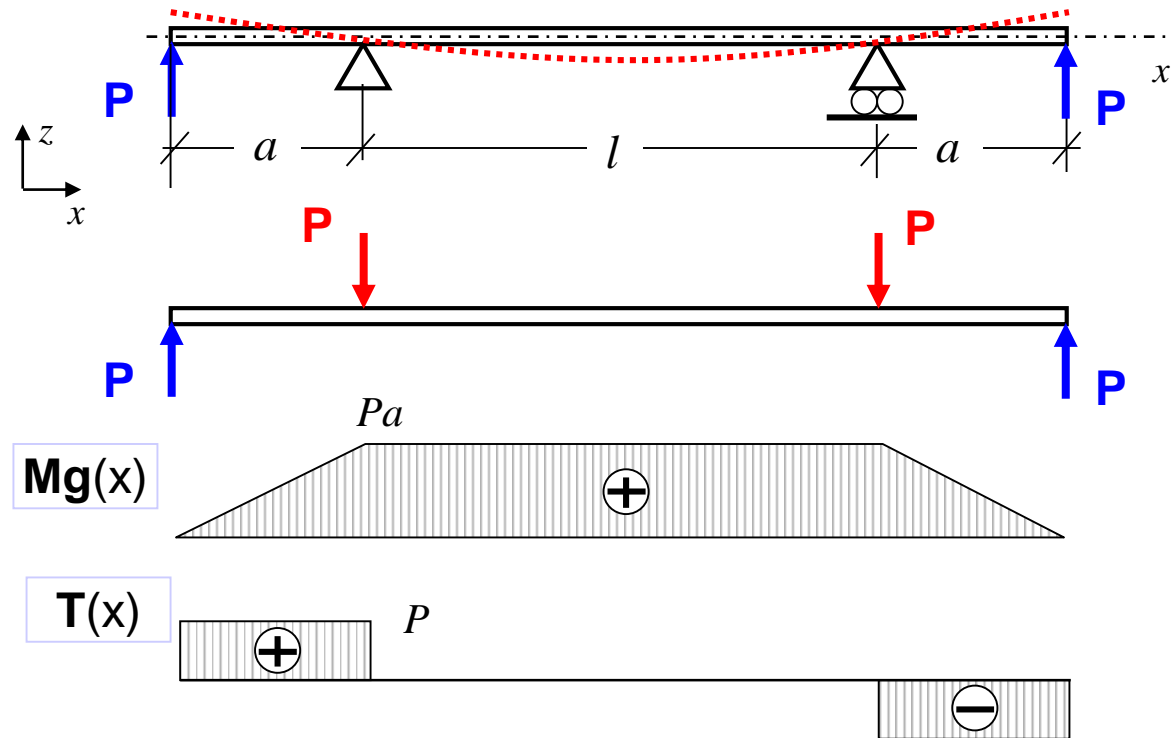
## Wykład 10

# Pręty zginane – belki

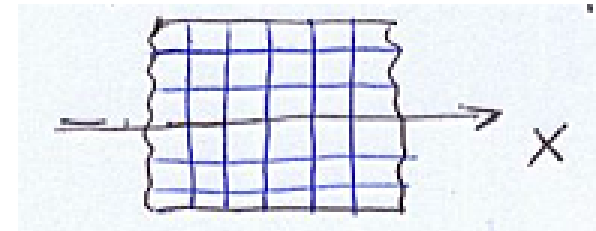
## Wyznaczanie stanu naprężenia

# Stan naprężenia w belce zginanej

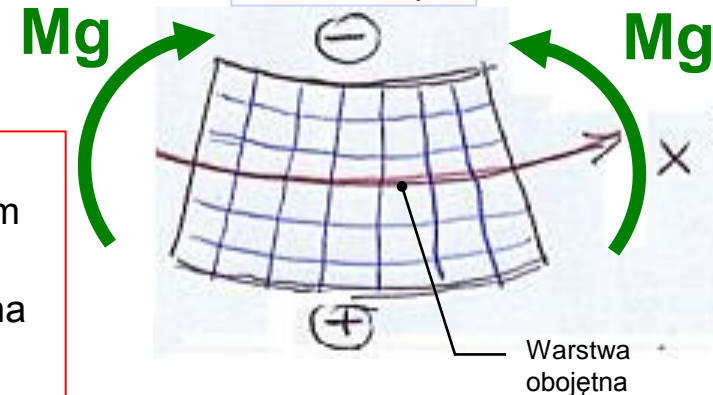
Czyste zginanie belki pryzmatycznej



Przed deformacją



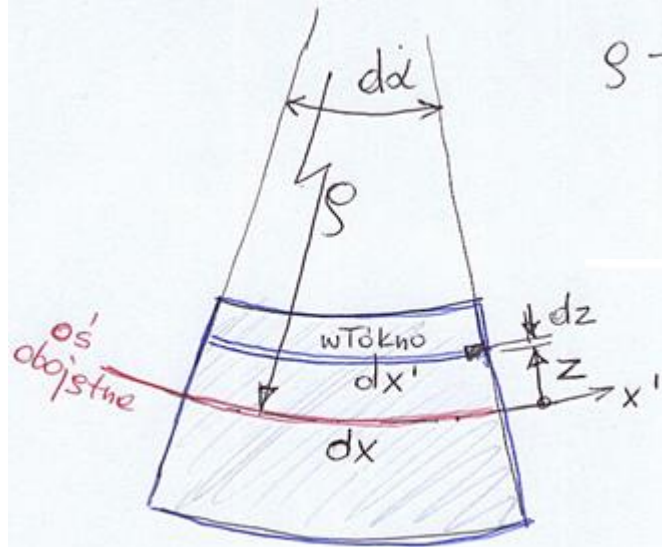
Po deformacji



**Założenia:**

- 1) przekroje płaskie, prostopadłe do osi belki przed odkształceniem muszą zostać płaskie i prostopadłe do zakrzywionej osi belki
- 2) Naciski (naprężenia) w kierunku poprzecznym do włókien można zaniedbać
- 3) Nie ma naprężeń tnących między włóknami

# Stan deformacji



$\rho$  - promień krzywizny

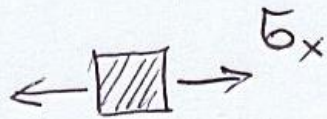
$$dx = \frac{dx'}{\rho - z} = \frac{dx}{\rho}$$

$$\frac{dx'}{dx} = \frac{\rho - z}{\rho}$$

$$\varepsilon_x(z) = \frac{dx' - dx}{dx} = \frac{\rho - z}{\rho} - \frac{\rho}{\rho}$$

$$\varepsilon_x(z) = -\frac{z}{\rho}$$

Prawo Hooke'a:

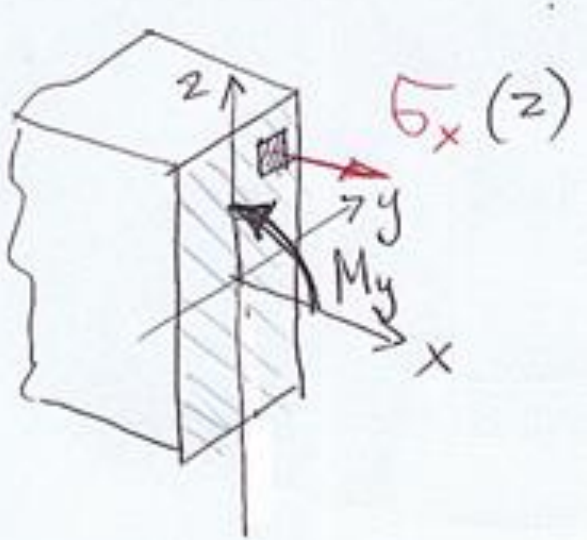


$$\varepsilon_x = \frac{1}{E} (\sigma_x - \nu (\sigma_y'' + \sigma_z''))$$

$$\varepsilon_x = \frac{\sigma_x}{E}$$

$$\sigma_x = -\frac{Ez}{\rho}$$

# Naprężenia normalne



1)  $N=0$  (brak rozciągania)

$$N = \int_A \sigma_x dA = -\frac{E}{\rho} \left( \int_A z dA \right) = 0$$

$S_y^c = 0$   
oś y jest centralna!

2)  $M_z=0$  (brak zginania wzgl. osi z)

$$M_z = \int_A y \cdot \sigma_x dA = -\frac{E}{\rho} \left( \int_A yz dA \right) = 0$$

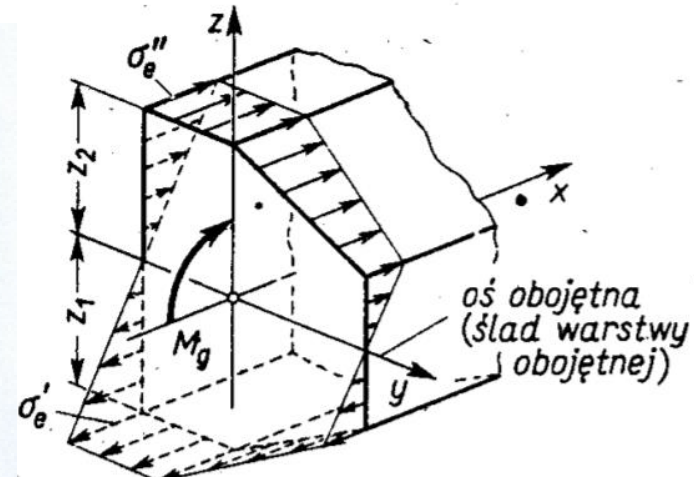
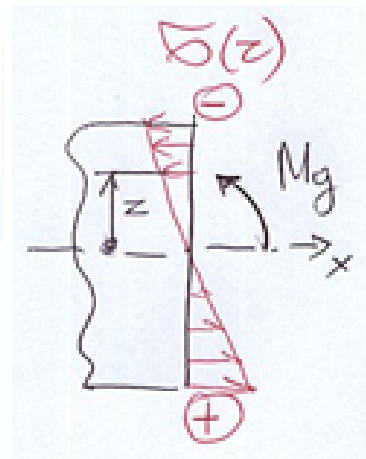
$J_{yz} = 0$  oś y jest główna!

3)  $M_y \neq 0$

$$M_y = -\int_A z \sigma_x dA = +\frac{E}{\rho} \left( \int_A z^2 dA \right) = J_y$$

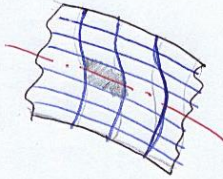
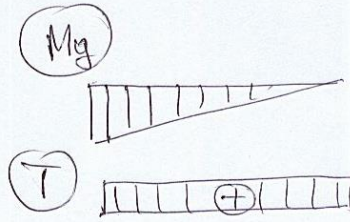
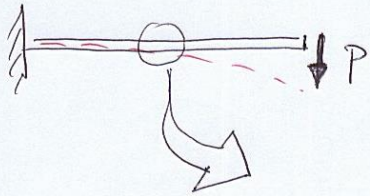
$$\frac{1}{\rho} = \frac{M_y}{E J_y}$$

$$\sigma(z) = -\frac{E z}{\rho} = -\frac{M_y}{J_y} \cdot z$$



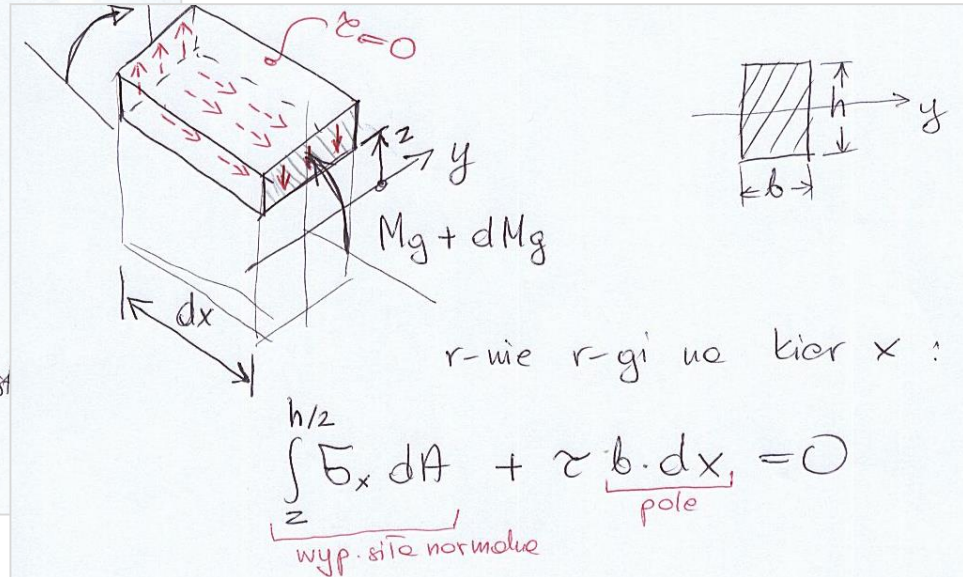
# Naprężenia tnące

Zginanie poprzeczne ( $T \neq 0$ )



Mg |

deformacje postaci



$$\int_z^{h/2} -\frac{dM_g}{J_y} \cdot z \cdot dA + \tau \cdot b \cdot dx = 0 \quad / : (dx \cdot b)$$

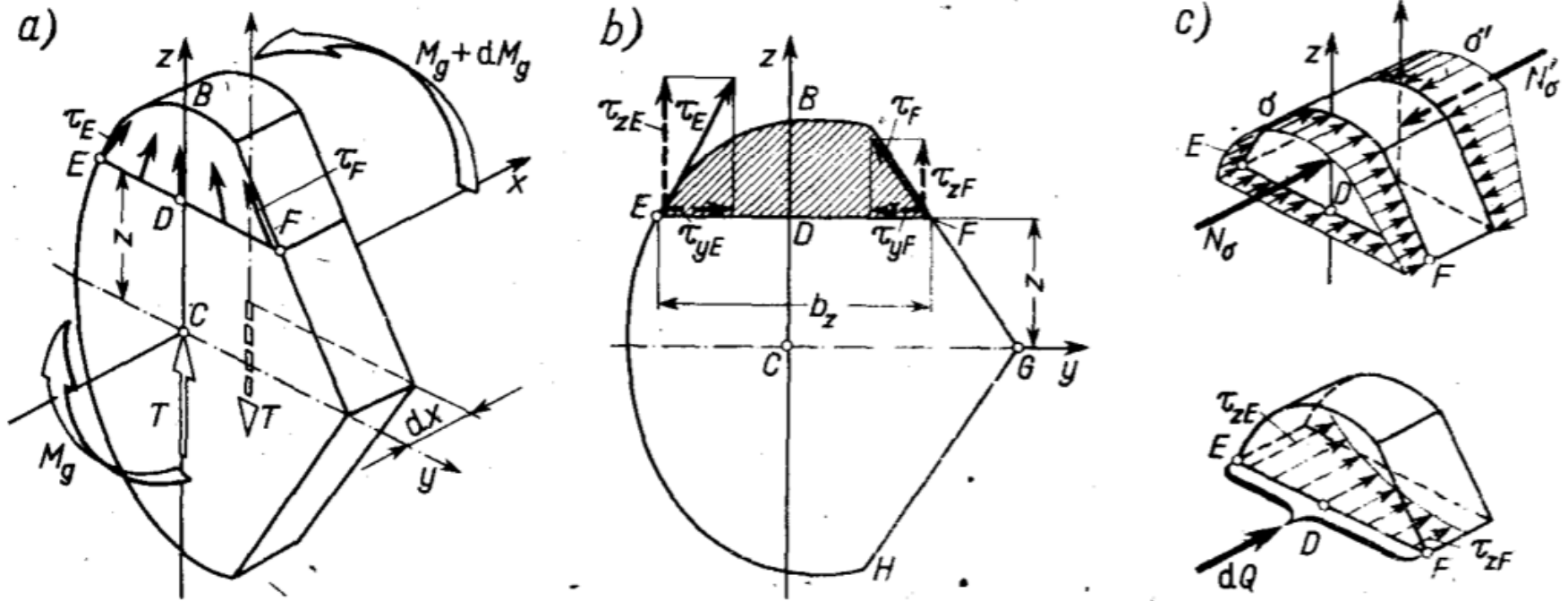
$$-\frac{dM_g}{dx} \cdot \frac{1}{b \cdot J_y} \cdot \int_z^{h/2} z \cdot dA + \tau = 0$$

" $S_y$  (moment statyczny)"

$T(x)$

$$\tau = \frac{T(x) \cdot S_y(z)}{J_y \cdot b}$$

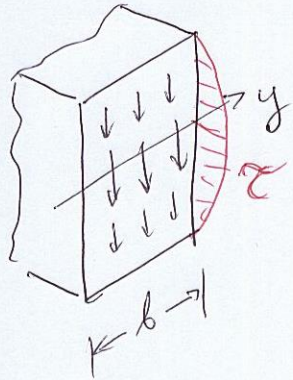
# Naprężenia tnące



Rys. 5.39. Określenie naprężeń  $\tau$  w belce o przekroju zwartym

# Przykład 10.1

dla prętkoju prostokątnego ( $b \times h$ )

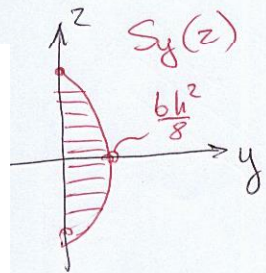
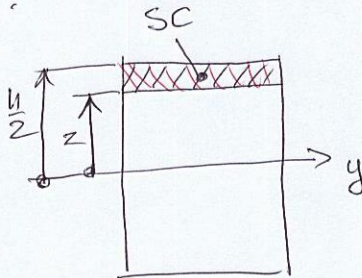


$$\tau = \frac{T \cdot S_y}{J_y \cdot b}$$

$$J_y = \frac{bh^3}{12}$$

$$S_y = \int_{-z}^{h/2} z \, dA$$

$$S_y = \underbrace{\left(\frac{h}{2} + z\right) \cdot \frac{1}{2}}_{\text{wsp. SC.}} \cdot \underbrace{\left(\frac{h}{2} - z\right) \cdot b}_{\text{pole}}$$



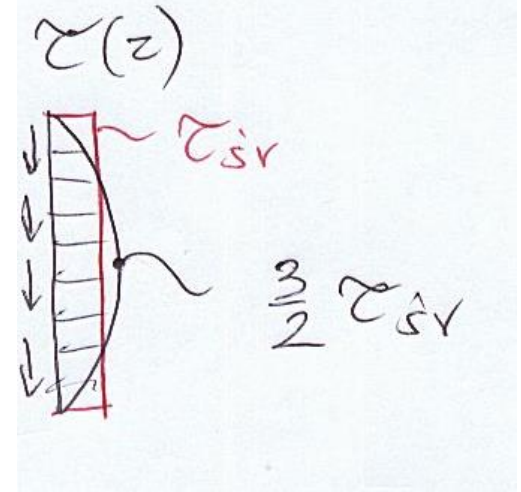
$$S_y = \frac{1}{2} \left( \frac{h^2}{4} - z^2 \right) \cdot b$$

moment statyczny  
części przekroju

$$S_y(z = \pm h/2) = 0$$

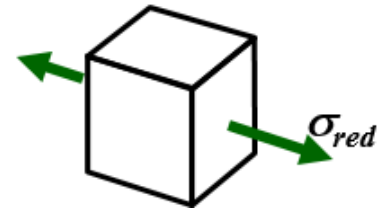
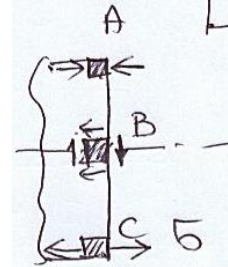
$$S_y(z = 0) = \frac{b \cdot h}{2} \cdot \frac{h}{4} = \frac{bh^2}{8}$$

$$\tau_{\max} = \tau(z=0) = \frac{T \cdot \frac{b \cdot h^2}{8}}{\frac{bh^3}{12} \cdot b} = \frac{3T}{2bh} = \frac{3}{2} \cdot \tau_{sr}$$



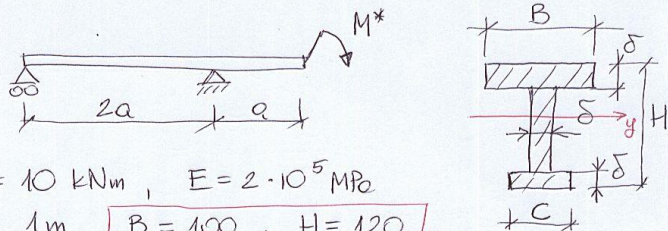
W belkach mamy zwykle

$$\tau \ll \sigma$$



# Przykład 10.2

## Zadanie

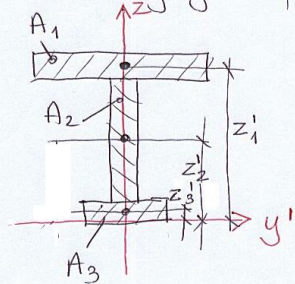


Dane:  $M^* = 10 \text{ kNm}$ ,  $E = 2 \cdot 10^5 \text{ MPa}$

$$a = 1 \text{ m}, \quad \begin{cases} B = 100 \\ C = 50 \end{cases}, \quad \begin{cases} H = 120 \\ \delta = 10 \end{cases}$$

Wyznaczyć:  $M_g(x)$ ,  $T(x)$ ,  $w(x)$  - linia ugięcia  
 $(\sigma_{\text{red}})_{\text{max}} = ?$

## I Charakterystyki linii przekroju



- środek ciężkości przekroju

$$\begin{cases} A_1 = B \cdot \delta = 10 \text{ cm} \cdot 1 \text{ cm} = 10 \text{ cm}^2 \\ z'_1 = H - \delta/2 = 12 - 0.5 = 11.5 \text{ cm} \end{cases}$$

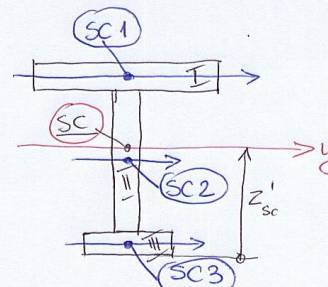
$$\begin{cases} A_2 = (H - 2\delta) \cdot \delta = 10 \cdot 1 = 10 \text{ cm}^2 \\ z'_2 = H/2 = 6 \text{ cm} \end{cases}$$

$$\begin{cases} A_3 = C \cdot \delta = 5 \cdot 1 = 5 \text{ cm}^2 \\ z'_3 = \delta/2 = 0.5 \text{ cm} \end{cases}$$

$$z'_{sc} = \frac{1}{A_1 + A_2 + A_3} (A_1 \cdot z'_1 + A_2 \cdot z'_2 + A_3 \cdot z'_3)$$

$$z'_{sc} = \frac{1}{25 \text{ cm}^2} (10 \cdot 11.5 + 10 \cdot 6 + 5 \cdot 0.5) = 7.1 \text{ cm}$$

Nasze centralne mom. bezw.



$$J_1^{wI} = \frac{B \cdot \delta^3}{12} = 0.833 \text{ cm}^4$$

$$J_2^{wI} = \frac{\delta \cdot (H - 2\delta)^3}{12} = 83.3 \text{ cm}^4$$

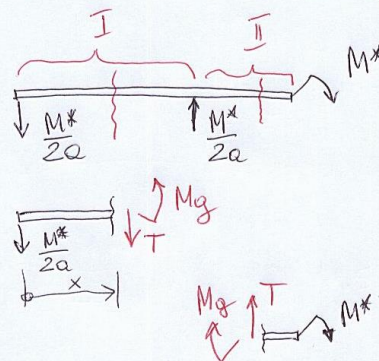
$$J_3^{wI} = \frac{C \cdot \delta^3}{12} = 0.417 \text{ cm}^4$$

$$J_y^{t.s} = J_1^{wI} + A_1 \cdot (z'_1 - z'_{sc})^2 + J_2^{wI} + A_2 \cdot (z'_2 - z'_{sc})^2 + J_3^{wI} + A_3 \cdot (z'_3 - z'_{sc})^2$$

$$J_y = 0.833 + 10 \cdot (11.5 - 7.1)^2 + 83.3 + 10 \cdot (6 - 7.1)^2 + 0.417 + 5 \cdot (0.5 - 7.1)^2 = 194.43 + 95.4 + 218.22$$

$$J_y = 508 \text{ cm}^4 \quad \text{moment bezwładności całego przekroju}$$

## II Wyznaczenie sił wewnętrznych



I  $x \in (0, 2a)$

$$T = -\frac{M^*}{2a} = -\frac{10}{2 \cdot 1} = -5 \text{ (kN)}$$

$$M_g = -\frac{M^*}{2a} \cdot x = -5x \text{ (kNm)}$$

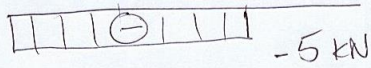
II  $x \in (2a, 3a)$

$$T = 0$$

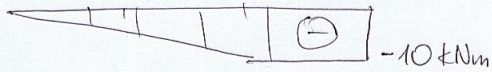
$$M_g = -M^* = -10 \text{ (kNm)}$$



T(x)



M<sub>g</sub>(x)



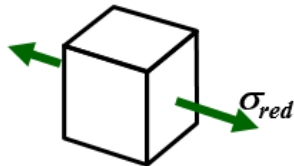
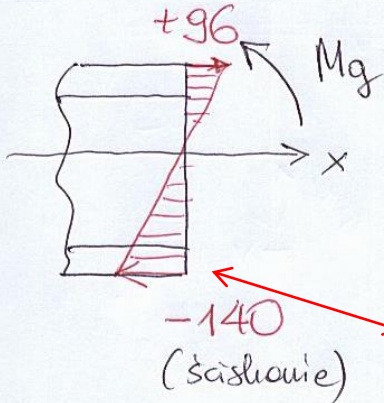
### III Naprężenia normalne

$$\sigma^G = -\frac{M_g(x)}{J_y} \cdot z_{\max} = -\frac{(-10 \cdot 10^3 \text{ Nm})}{508 \cdot 10^{-8} \text{ m}^4} \cdot 4.9 \cdot 10^{-2} \text{ m}$$

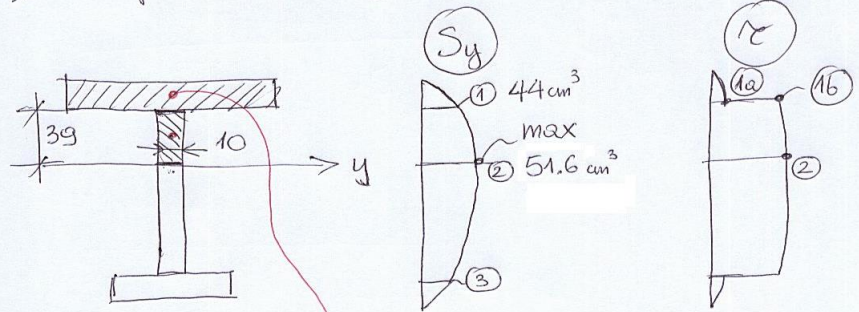
$$\boxed{\sigma^G = 96 \cdot 10^6 \frac{\text{N}}{\text{m}^2} = 96 \text{ MPa}}$$

$$\sigma^D = -\frac{M_g(x)}{J_y} \cdot z_{\min} = -\frac{(-10 \cdot 10^3 \text{ Nm})}{508 \cdot 10^{-8} \text{ m}^4} \cdot (-7.1 \cdot 10^{-2} \text{ m})$$

$$\boxed{\sigma^D = -140 \cdot 10^6 \frac{\text{N}}{\text{m}^2} = -140 \text{ MPa}}$$



### IV Naprężenia tnące



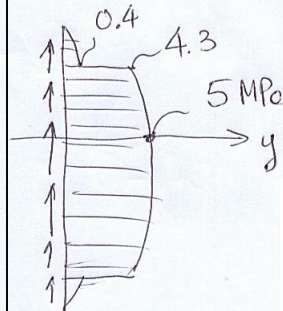
$$S_y^{(1)} = B \cdot \delta \cdot (z'_1 - z'_{sc}) = \int_0^{z_{\max}} z \, dA = 44 \text{ cm}^3$$

$$S_y^{(2)} = \frac{B \cdot \delta \cdot (z'_1 - z'_{sc})^2}{44 \text{ cm}^3} + \frac{3.9 \cdot 1}{2} \cdot \frac{3.9}{2} = 51.6 \text{ cm}^3$$

$$\tau^{(1a)} = \frac{T \cdot S_y^{(1)}}{J_y \cdot B} = \frac{-5 \cdot 10^3 \text{ N} \cdot 44 \text{ cm}^3}{508 \text{ cm}^4 \cdot 10 \text{ cm}} = -43 \frac{\text{N}}{\text{cm}^2} = -0.43 \text{ MPa}$$

$$\tau^{(1b)} = \frac{T \cdot S_y^{(1)}}{J_y \cdot \delta} = \frac{-5 \cdot 10^3 \text{ N} \cdot 44 \text{ cm}^3}{508 \text{ cm}^4 \cdot 1 \text{ cm}} = -430 \frac{\text{N}}{\text{cm}^2} = -4.3 \text{ MPa}$$

$$\tau^{(2)} = \frac{T \cdot S_y^{(2)}}{J_y \cdot \delta} = \frac{-5 \cdot 10^3 \text{ N} \cdot 51.6 \text{ cm}^3}{508 \text{ cm}^4 \cdot 1 \text{ cm}} = -508 \frac{\text{N}}{\text{cm}^2} = -5 \text{ MPa}$$

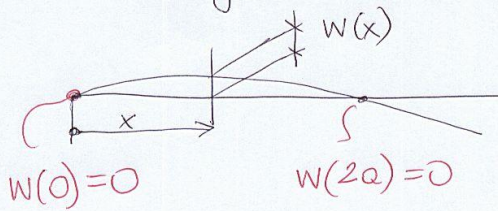


szacowanie napr. tnących

$$\tau_{sr} \approx \frac{T}{A_2} = \frac{5000 \text{ N}}{1000 \text{ mm}^2} = 5 \text{ MPa}$$

↑  
pole środkowe

V Linia ugięcia



$$w'' \approx \frac{1}{S} = \frac{M_g(x)}{EJ_y}$$

$x \in (0, 2a)$

$$w_{I}'' = \frac{M_{gI}}{EJ_y} = \left[ -\frac{M^* \cdot x}{2aEJ_y} \right]$$

$$w_{I}' = -\frac{M^* x^2}{4aEJ_y} + C_I$$

$$w_I = -\frac{M^* x^3}{12aEJ_y} + C_I x + D_I$$

warunki brzegowe:

$$w_I(0) = D_I = 0$$

$$w_I(2a) = \frac{-M^*(2a)^3}{12aEJ_y} + C_I \cdot 2a = 0$$

warunki ciągłości:

$$w_I(2a) = w_{II}(2a) = -\frac{M^*(2a)^2}{2EJ_y} + C_{II} \cdot 2a + D_{II} = 0$$

$$w_I'(2a) = -\frac{M^* \cdot (2a)^2}{4aEJ_y} + C_I = w_{II}'(2a) = -\frac{M^* \cdot 2a}{EJ_y} + C_{II}$$

$x \in (2a, 3a)$

$$w_{II}'' = \frac{M_{gII}}{EJ_y} = \left[ -\frac{M^*}{EJ_y} \right]$$

$$w_{II}' = -\frac{M^* x}{EJ_y} + C_{II}$$

$$w_{II} = -\frac{M^* x^2}{2EJ_y} + C_{II} x + D_{II}$$

$$C_I = \frac{M^* a}{3EJ} = \frac{10 \cdot 10^3 \text{ Nm} \cdot 1 \text{ m}}{3 \cdot 2 \cdot 10^{11} \frac{\text{N}}{\text{m}^2} \cdot 508 \cdot 10^{-8} \text{ m}^4} = 0,00328 \text{ rad} \quad (0,19^\circ)$$

$$D_I = 0$$

$$C_{II} = \frac{4}{3} \frac{M^* a}{EJ} = 0,0131 \text{ rad} \quad (0,752^\circ)$$

$$D_{II} = -\frac{2}{3} \frac{M^* a^2}{EJ} = -0,0066 \text{ m} = -6,6 \text{ mm}$$

$$w_I = \frac{M^*}{12aEJ} (4a^2 - x^2) \cdot x$$

$$w_I' = \frac{M^*}{12aEJ} (4a^2 - 3x^2)$$

$$w_{II} = \frac{M^*}{6EJ} (2a-x)(3x-2a)$$

$$w_{II}' = \frac{M^*}{3EJ} (4a-3x)$$

